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STATISTICAL THEORY AND METHOD

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Short Title:
Statistical Theory Abstracts

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INTERNATIONAL JOURNAL OF ABSTRACTS STATISTICAL THEORY AND METHOD

COVERAGE OF JOURNAL

THE aim of this journal of abstracts is to give complete coverage of papers in the field of statistical theory and new contributions to statistical method. Papers which only report applications or examples of existing statistical theory and method will not be included. There are approximately two hundred and fifty journals published in various parts of the world which are wholly or partly devoted to the field of statistical theory and method and which will be brought within the scope of this journal of abstracts. In due course it will be possible to issue a complete list of these journals. In the case of the following journals, however, being those which are wholly devoted to statistical theory—all contributions, whether a paper, note or miscellanea, will be abstracted :

Annals of Mathematical Statistics,
Biometrika
Journal, Royal Statistical Society (Series B)
Bulletin of Mathematical Statistics
Annals, Institute of Statistical Mathematics

Within the larger group of journals, which are not wholly devoted to statistical theory and method, there are some journals which have the vast majority of their contributions in this field. These journals, therefore, will be abstracted on a virtually complete basis :

Biometrics
Metrika
Metron
Review, International Statistical Institute
Technometrics
Sankhyā

After experience in the publication of this journal of abstracts it may be found desirable to add further journals to these lists. In any case readers may be assured that all papers properly to be included in this journal of abstracts will be included irrespective of such notification. If any reader of this journal discovers a paper which happens to have been overlooked, the General Editor will be pleased to be informed so that the appropriate abstract can be made : always provided that the date of publication is after 1st October 1958, when the abstracting for this journal commenced.

In addition to the ordinary journals, there are two kinds of publication which fall within the scope of this journal of abstracts. They are the experiment and other research station reports—which occur particularly in the North American region—and the reports of conferences, symposia and seminars. Whilst these latter may be included in the book review sections of journals it is unusual for any individual contribution to be noted at any length. These publications are, in effect, special collections of papers and for this reason the appropriate arrangements will be made for them to be included in this journal. By the same token, abstracts of papers given at conferences and reproduced in an appropriate journal will be disregarded until the definitive publication is available.

FORMAT OF JOURNAL

The abstracts will be about 400 words long—the recommendation of UNESCO for the " long " abstract service : they will be in the English language although the language of the original paper will be indicated on the abstract together with the name of the abstractor. In addition, the address of the author(s) will be given in sufficient detail to facilitate contact in order to obtain further detail or request an offprint. Suitable indexes will be provided annually in a supplement.

The scheme of classification has been developed upon lines that will facilitate the transfer to punched cards of the code numbers allocated to each abstract: to allow for future development it is suggested that use is made of 4-column fields. Each abstract will have two classification numbers: the primary number in heavy type to indicate the basic topic of the paper and the secondary number in brackets to take account of the most important cross-reference. The sheets of the journal are colour-coded according to the twelve main sections of the classification and it should be noted that it is the main section number of the primary classification which determines the colour code for each abstract. It is believed that this method of colour-coding the pages will provide a distinctive visual aid to the identification of abstracts both when the journal is in bound form or dismantled and filed on cards or in binders. The format and simple binding allows of the following alternative treatments by users of the journal:

(*a*) Leave intact as a shelf-periodical.

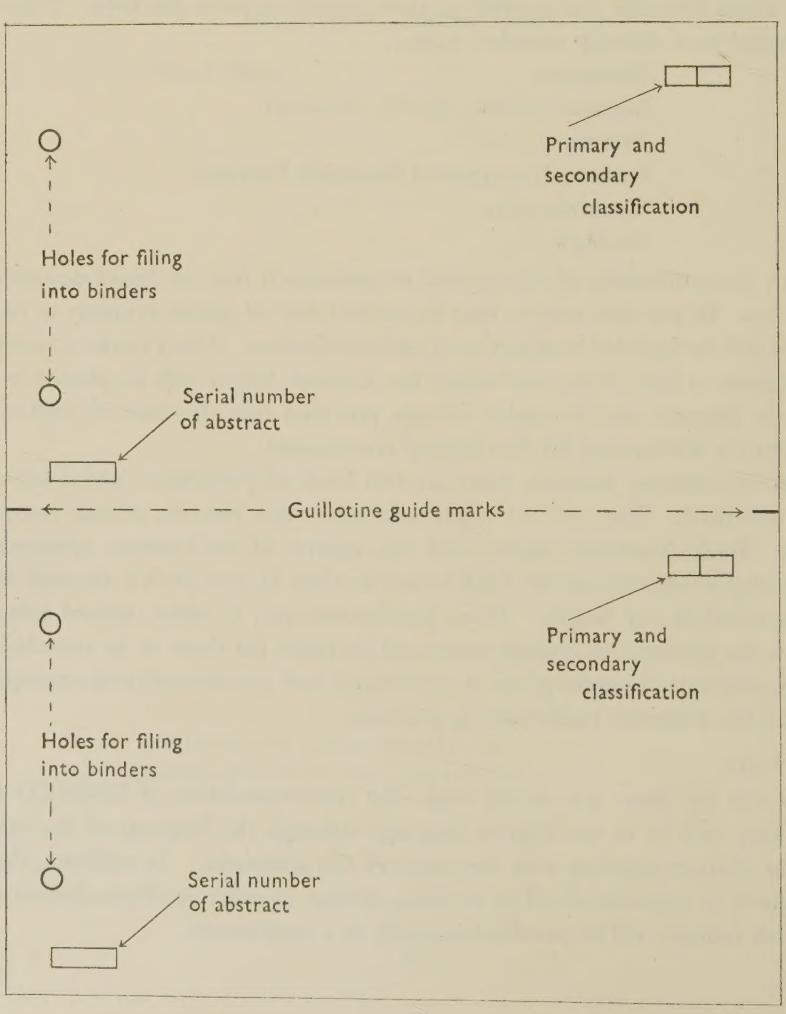
(*b*) Split and filed in page form according to the main sections of the classification.

(*c*) Split and guillotine (single cut) each page ready for:

(i) pasting onto standard index cards.

(ii) filing in loose-leaf or other binders—for which the appropriate holes are punched.

It will also be possible for those users who need a completely referenced file to insert skeleton cards or sheets according to the secondary classification number provided on each abstract.


Primary and secondary classification
Holes for filing into binders
Serial number of abstract
Guillotine guide marks
Primary and secondary classification
Holes for filing into binders
Serial number of abstract

INTERNATIONAL JOURNAL OF ABSTRACTS STATISTICAL THEORY AND METHOD

LIST OF PUBLICATIONS FOR REGULAR SCANNING

The following list of publications in the statistical and allied fields will be regularly examined for papers which should be abstracted for this Journal. Other journals will also be scrutinised from time to time and a check maintained by scanning other relevant abstracting journals. In addition, abstracts will be included of special collections of papers as published in reports of conferences, symposia and seminars together with the published reports of experiment and other research stations. The coverage of this Journal amounts to at least 400 sources of publication

Acta Acad. Cient., Lima	Acta de la Academia Cientifica, Lima	Peru
Acta Math., Stockh.	Acta Mathematica	Sweden
Acta Math. Acad. Sci. Hung.	Acta Mathematica Academiae Scientiarum Hungaricae	Hungary
Acta Math. Sinica	Acta Mathematica Sinica	China
Acta Pontif. Acad. Sci.	Acta Pontificiae Academiae Scientiarum	Italy
Advanced Management	Advanced Management	U.S.A.
Advanc. Enzymol.	Advances in Enzymology and Related Subjects	U.S.A.
Advanc. Protochem.	Advances in Protochemistry	U.S.A.
Advanc. Vet. Sci.	Advances in Veterinary Science	U.S.A.
Agric. Engng., St Joseph, Mich.	Agricultural Engineering	U.S.A.
Agriculture, Lond.	Agriculture (Min. of Agric.)	Great Britain
Agron. J.	Agronomy Journal	U.S.A.
Allg. Statist. Arch.	Allgemeines Statistisches Archiv	Germany
Amer. Econ. Rev.	American Economic Review	U.S.A.
Amer. Heart J.	American Heart Journal	U.S.A.
Amer. J. Clin. Nutr.	American Journal of Clinical Nutrition	U.S.A.
Amer. J. Clin. Path.	American Journal of Clinical Pathology	U.S.A.
Amer. J. Digestive Diseases	American Journal of Digestive Diseases	U.S.A.
Amer. J. Hum. Genet.	American Journal of Human Genetics	U.S.A.
Amer. J. Hyg.	American Journal of Hygiene	U.S.A.
Amer. J. Med.	American Journal of Medicine	U.S.A.
Amer. J. Physiol.	American Journal of Physiology	U.S.A.
Amer. J. Publ. Health	American Journal of Public Health	U.S.A.
Amer. J. Sci.	American Journal of Science	U.S.A.
Amer. J. Sociol.	American Journal of Sociology	U.S.A.
Amer. J. Vet. Res.	American Journal of Veterinary Research	U.S.A.
Amer. Nat.	American Naturalist	U.S.A.
Amer. Statistician	American Statistician	U.S.A.
Analyt. Chem.	Analytical Chemistry	U.S.A.
An. Soc. Cient. Argent.	Anales de la Sociedad Cientifica Argentina	Argentina
Ann. Acad. Sci. Fenn.	Annales Academiae Scientiarum Fennicae	Finland
Ann. Appl. Biol.	Annals of Applied Biology	Great Britain
Ann. Fac. Econ. Com. Catania	Annali della Facoltà di Economia e Commercio, Catania (University)	Italy
Ann. Fac. Econ. Com. Palermo	Annali della Facoltà di Economia e Commercio, Palermo (University)	Italy
Ann. Fac. Sci. Porto	Annaes da Faculdade de Ciências do Porto	Portugal
Ann. Hum. Genet.	Annals of Human Genetics (London)	Great Britain
Ann. Inst. Poincaré	Annales de l'Institut Henri Poincaré	France
Ann. Inst. Statist. Bari	Annali dell'Instituto di Statistica, Bari (University)	Italy
Ann. Inst. Statist. Math., Tokyo	Annals of the Institute of Statistical Mathematics	Japan
Ann. Inst. Sup. Sci. Econ. Financ.	Annaes do Instituto Superior de Ciências Economicas e Financieras	Portugal
Ann. Math. Statist.	Annals of Mathematical Statistics	U.S.A.
Ann. N.Y. Acad. Sci.	Annals of the New York Academy of Science	U.S.A.
Ann. Sci. Éc. Norm. Sup., Paris	Annales Scièntifiques de l'École Normale Supérieure	France
Ann. Sci. Écon. Appl.	Annales de Sciences Économique Appliquées	France
Ann. Soc. Sci. Bruxelles, I	Annales de la Société Scientifique de Bruxelles—Serie I	Belgium

Ann. Statist., Rome	Annali de Statistica (Instituto Centrale di Statistica, Rome)	Italy
Ann. Univ. Lyon, A	Annales de l'Université de Lyon—Section A	France
Ann. Univ. Sci. Budapest. (Sect. Math.)	Annales Universitatis Scientiarum Budapestiensis de Rolando Eötvös nominatae—Sectio Mathematica	Hungary
Annu. Rev. Physiol.	Annual Review of Physiology	U.S.A.
Aplik. Mat.	Aplikace Matematiky	Czechoslovakia
Appl. Sci. Res., Hague	Applied Scientific Research, The Hague	Netherlands
Appl. Statist.	Applied Statistics	Great Britain
Archimede	Archimede	Italy
Arch. Biochem. Biophys.	Archives of Biochemistry and Biophysics	U.S.A.
Arch. Math., Karlsruhe	Archiv der Mathematik	Germany
Ark. Mat.	Arkiv för Matematik	Sweden
Atti (mem.) Accad. Lincei	Atti della Accademia Nazionale dei Lincei	Italy
Atti Riun. Sci. Soc. Ital. Statist.	Atti della Riunione Scientifica Società Italiana di Statistica, Rome	Italy
Aust. J. Agric. Res.	Australian Journal of Agricultural Research	Australia
Aust. J. Appl. Sci.	Australian Journal of Applied Science	Australia
Aust. J. Biol. Sci.	Australian Journal of Biological Sciences	Australia
Aust. J. Phys.	Australian Journal of Physics	Australia
Aust. J. Sci.	Australian Journal of Science	Australia
Aust. J. Statist.	Australian Journal of Statistics	Australia
Beil. Mber. Öst. Inst. Wirtschaftsf.	Beilagen zu den Monatsberichten des Österreichischen Institutes für Wirtschaftsforschung	Austria
Bell Lab. Record	Bell Laboratories Record	U.S.A.
Bell Syst. Monographs	Bell System Monographs	U.S.A.
Bell Syst. Tech. J.	Bell System Technical Journal	U.S.A.
Biometrics	Biometrics	U.S.A.
Biometrika	Biometrika	Great Britain
Biom. Zeit.	Biometrische Zeitschrift	Germany
Blä. Dtsch. Ges. Versich.-math.	Blätter der Deutschen Gesellschaft für Versicherungsmathematik	Germany
Bol. Fac. Filos. Ciênt., S. Paulo	Boletim da Faculdade de Filosofía, Ciências e Letras, São Paulo	Brazil
Bol. Inst. Actuar. Port.	Boletim do Instituto dos Actuarios Portugueses	Portugal
Bol. Soc. Mat. Mexicana	Boletim de la Sociedad Matemática Mexicana	Mexico
Boll. Cent. Ric. Operat.	Bollettino del Centro per la Ricerca Operativa	Italy
Boll. Un. Mat. Ital.	Bollettino dell'Unione Matematica Italiana	Italy
Brit. J. Nutr.	British Journal of Nutrition	Great Britain
Brit. J. Prev. Soc. Med.	British Journal of Preventive and Social Medicine	Great Britain
Brit. J. Sociol.	British Journal of Sociology	Great Britain
Brit. J. Statist. Psychol.	British Journal of Statistical Psychology	Great Britain
Bull. Acad. Sci., III	Bulletin de l'Académie des Sciences (Classe 3)	Poland
Bull. Amer. Soc. Test. Mat.	Bulletin, American Society for Testing Materials	U.S.A.
Bull. Ass. Actuair. Suisses	Bulletin de l'Association des Actuaires Suisses	Switzerland
Bull. Calcutta Math. Soc.	Bulletin of the Calcutta Mathematical Society	India
Bull. Calcutta Statist. Ass.	Bulletin, Calcutta Statistical Association	India
Bull., C.S.I.R.O., Aust.	Bulletins, Commonwealth Scientific and Industrial Research Organisation	Australia
Bull. Indian Soc. Qual. Contr.	Bulletin, Indian Society of Quality Control	India
Bull. Indian Stand. Inst.	Bulletin, Indian Standards Institution	India
Bull. Inst. Égypte	Bulletin de l'Institut d'Égypte, Cairo	Egypt
Bull. Jap. Statist. Soc.	Bulleting of the Japan Statistical Society	Japan
Bull. Math. Statist.	Bulletin of Mathematical Statistics, Research Association of Statistical Sciences	Japan
Bull. Nat. Inst. Sci. India	Bulletin, National Institute of Sciences of India	India
Bull. Oxf. Univ. Inst. Statist.	Bulletin of the Oxford University Institute of Statistics	Great Britain
Bull. Res. Inst. Kerala	Bulletin, Central Research Institute, Kerala (University)	India
Bull. Sci. Math.	Bulletin des Sciences Mathématique	France
Bull. Soc. Math. Belg.	Bulletin de la Société Mathématique de Belgique, Gembloux	Belgium

Bull. Soc. Math. France	Bulletin de la Société Mathématique de France	France
Bull. Statist. Soc., N.S.W.	Bulletin of the Statistical Society, N.S. Wales	Australia
Canad. J. Econ. Polit. Sci.	Canadian Journal of Economic and Political Science	Canada
Cancer, New York	Cancer (New York)	U.S.A.
Cas. pěst. math.	Casopis pro pěstování mathematiky	Czechoslovakia
Chem. Engng. Progr.	Chemical Engineering Progress	U.S.A.
Clin. Chem.	Clinical Chemistry	U.S.A.
Colect. Estud.	Colectanea de Estudos	Spain
Coll. Mat.	Collectanea Matematica	Spain
Colloq. Math.	Colloquium Mathematicum	Poland
Comment. Math. Helvetia	Commentarii Mathematica Helvetia	Switzerland
Comment. Pontif. Acad. Sci.	Commentationes Pontificiae Academiae Scientiarum	Italy
Compos. Math., Groningen	Compositio Mathematica, Groningen	Netherlands
C.R. Acad. Sci., Paris	Comptes Rendus de l'Académie des Sciences, Paris	France
Cornell Vet.	Cornell Veterinarian	U.S.A.
Current Science	Current Science	India
Curso Mira Fernandes	Curso Mira Fernandes	Portugal
Czech. Math. J.	Czechoslowak Mathematical Journal	Czechoslovakia
Dacca Univ. Statist. Bull.	Dacca University Statistical Bulletin	Pakistan
Defense Sci. J.	Defense Science Journal	India
Dokl. Akad. Nauk. SSSR	Doklady Akademiia Nauk, SSSR.	U.S.S.R.
Dopov. Akad. Nauk. Ukrain, SSR	Dopovedy Akademiia Nauk Ukrainskoj, SSR	U.S.S.R.
Écon. Appl.	Économie Appliquée	France
Econ. Internatzionale	Economia Internatzionale	Italy
Econ. J.	Economic Journal	Great Britain
Écon. Leban. Arabe	L'Économie Lebanaise & Arabe, Beirut	Lebanon
Econ. Record	Economic Record	Australia
Econometrica	Econometrica	U.S.A.
Economica	Economica	Great Britain
Edinb. Math. Notes	Edinburgh Mathematical Notes	Great Britain
Égypte Contemp.	L'Égypte Contemporaine, Cairo	Egypt
Egyptian Statist. J.	Egyptian Statistical Journal	Egypt
Ekon. Tidskr.	Ekonomisk Tidskrifts	Sweden
Emp. J. Exp. Agric.	Empire Journal of Experimental Agriculture	Great Britain
Endocrinology	Endocrinology	U.S.A.
Engineer, Lond.	Engineer	Great Britain
Engineering, Lond.	Engineering	Great Britain
Ergebn. Naturw. Grenzgeb.	Ergebnisse der exakten Naturwissenschaften	Germany
Esta	Esta	Spain
Estadistica	Estadistica	U.S.A.
Euclides, Groningen	Euclides, Groningen	Netherlands
Eugen. Rev.	Eugenics Review	Great Britain
Evolution	Evolution	U.S.A.
Food Res.	Food Research	U.S.A.
Food Tech.	Food Technology	U.S.A.
Ganita	Ganita, Lucknow-Allahabad (Universities)	India
Gaz. Mat.	Gazeta de Matemática	Portugal
G. Economisti	Giornale degli Economisti e Annali di Economia	Italy
G. Ist. Ital. Attuari	Giornale dell'Istituto Italiano degli Attuari	Italy
Genetics	Genetics	U.S.A.
Genus	Genus	Italy
Growth	Growth	U.S.A.
Harvard Busin. Rev.	Harvard Business Review	U.S.A.
Helvetica Phys. Acta	Helvetica Physica Acta	Switzerland

Heredity	Heredity	Great Britain
Hum. Biol.	Human Biology	U.S.A.
IFO-Studien	IFO-Studien	Germany
Inc. Statist.	Incorporated Statistician	Great Britain
Indag. Math.	Indagationes Mathematicae, Amsterdam	Netherlands
Indian Coun. Agric. Res. Bull.	Indian Council of Agricultural Research : Bulletin	India
Indian Coun. Agric. Res. Statist. Newsletter	Indian Council of Agricultural Research : Statistical Newsletter	India
Indian Econ. J.	Indian Economic Journal	India
Indian Econ. Rev.	Indian Economic Review	India
Indian J. Agric. Econ.	Indian Journal of Agricultural Economics	India
Indian J. Phys.	Indian Journal of Physics	India
Indian J. Power & River Devel.	Indian Journal of Power and River Development	India
Indian J. Soc. Work	Indian Journal of Social Work	India
Indian Math. Soc. J.	Indian Mathematical Society Journal	India
Industr. Engng. Chem.	Industrial and Engineering Chemistry	U.S.A.
Industr. Qual. Contr.	Industrial Quality Control	U.S.A.
Industria, Milano	L'Industria	Italy
Iowa Acad. Sci. J.	Iowa Academy of Science Journal	U.S.A.
Iowa Exp. Sta. Publ.	Iowa State College Experiment Station Publications	U.S.A.
Izv. Akad. Nauk, SSSR	Izvestiya Akademiia Nauk, SSSR	U.S.S.R.
Izv. Akad. Nauk, Uzbek., SSR.	Izvestiya Akademiia Nauk, Uzbekskoj, SSR.	U.S.S.R.
Jb. Nat.-Ökon. Statist.	Jahrbücher für Nationalökonomie und Statistik	Germany
Jber. Dtsch. Mathver.	Jahresbericht der Deutschen Mathematikervereinigung	Germany
J. Agric. Food Chem.	Journal of Agricultural and Food Chemistry	U.S.A.
J. Agric. Sci.	Journal of Agricultural Science	Great Britain
J. Amer. Pharm. Ass.	Journal of the American Pharmaceutical Association (Scientific Edition)	U.S.A.
J. Amer. Statist. Ass.	Journal, American Statistical Association	U.S.A.
J. Amer. Vet. Med. Ass.	Journal of American Veterinary Medicine Association	U.S.A.
J. Animal Sci.	Journal of Animal Sciences	U.S.A.
J. Appl. Physiol.	Journal of Applied Physiology	U.S.A.
J. Ass. Comp. Mach.	Journal of the Association of Computing Machinery	U.S.A.
J. Aust. Inst. Agric. Sci.	Journal of the Australian Institute of Agricultural Science	Australia
J. Aust. Math. Soc.	Journal of the Australian Mathematical Society	Australia
J. Biol. Chem.	Journal of Biological Chemistry	U.S.A.
J. Chronic Diseases	Journal of Chronic Diseases	U.S.A.
J. Clin. Invest.	Journal of Clinical Investigation	U.S.A.
J. Dairy Sci.	Journal of Dairy Science	U.S.A.
J. Ecology	Journal of Ecology	Great Britain
J. Econ. Ent.	Journal of Economic Entomology	U.S.A.
J. Exp. Educ.	Journal of Experimental Education	U.S.A.
J. Exp. Med.	Journal of Experimental Medicine	U.S.A.
J. Forestry	Journal of Forestry	U.S.A.
J. Gen. Physiol.	Journal of General Physiology	U.S.A.
J. Genet.	Journal of Genetics	India
J. Heredity	Journal of Heredity	U.S.A.
J. Hyg., Camb.	Journal of Hygiene, Cambridge	Great Britain
J. Industr. Econ.	Journal of Industrial Economics	Great Britain
J. Industr. Engng.	Journal of Industrial Engineering	U.S.A.
J. Indian Soc. Agric. Statist.	Journal of the Indian Society of Agricultural Statistics	India
J. Inst. Actuar.	Journal of the Institute of Actuaries	Great Britain
J. Inst. Actuar. Stud. Soc.	Journal of the Institute of Actuaries Students' Society	Great Britain
J. Madras Univ., B	Journal of Madras University, Section B.	Great Britain
J. Mammalogy	Journal of Mammalogy	U.S.A.
J. Marine Res.	Journal of Marine Research	U.S.A.
J. Math. Pures Appl.	Journal de Mathématiques Pures et Appliquées	France
J. Math. Soc. Japan	Journal of the Mathematical Society of Japan	Japan
J. Nutrition	Journal of Nutrition	U.S.A.
J. Operat. Res.	Journal of the Operations Research Society of America	U.S.A.

J. Pharmacol.	Journal of Pharmacology and Experimental Therapeutics	U.S.A.
J. Plant Physiol.	Journal of Plant Physiology	U.S.A.
J. Polit. Econ.	Journal of Political Economy	U.S.A.
J. reine angew. Math.	Journal für reine und angewandte Mathematik	Germany
J. Res., Nat. Bur. Stand.	Journal of Research, National Bureau of Standards, Washington	U.S.A.
J. Roy. Soc., N.S.W.	Journal of the Royal Society, N.S. Wales	Australia
J. R. Statist. Soc. (A & B)	Journal of the Royal Statistical Society (Series A & B)	Great Britain
J. Sci. Engng.	Journal of Science and Engineering	India
J. Sci. Industr. Res.	Journal of Scientific and Industrial Research	U.S.A.
J. Sci. Industr. Res., India	Journal of Scientific and Industrial Research	India
J. Soc. Statist. Paris	Journal de la Société Statistique de Paris	France
J. Statist. Social Inquiry Soc.	Journal of the Statistical and Social Inquiry Society of Ireland	Ireland
J. Univ. Baroda	Journal, Maharaja Sayajiroa, Baroda University	India
Kodai (Tokyo Inst. Tech.)	Kodai (Mathematical Seminar reports) Tokyo Institute of Technology	Japan
Magy. Tud. Akad. III *Oszt. Közl.*	Magyar Tudományos Akadémia Matematika és Fizikai Osztályának Közleményei	Hungary
Management Sci.	Management Science	U.S.A.
Manager (India)	Manager	India
Manchester School	Manchester School	Great Britain
Matemática	Matemática, Notas del Instituto de Matemática Facultad de Ingeniería de Rosario	Argentine
Mat. Lapok	Matematikai Lapok	Hungary
Math. Ann.	Mathematische Annalen	Germany
Math. Gaz.	Mathematical Gazette	Great Britain
Math. Nachr.	Mathematische Nachrichten	Germany
Math. Stud.	Mathematics Students	India
Math. Tab., Wash.	Mathematical Tables and other aids to Computation	U.S.A.
Math., Tech., Wirtschaft	Mathematik, Technik, Wirtschaft (Zeitschrift für moderne Rechentechnik und Automation)	Austria
Math. Zeit.	Mathematische Zeitschrift	Germany
Meded. Landbhoogesch., Wageningen	Mededeelingen van de Landbouwhoogeschool, Wageningen	Netherlands
Mem. Coll. Sci., Kyoto	Memoirs of the College of Science, Kyoto (University)	Japan
Mem. Fac. Sci. A, *Kyusyu*	Memoirs of the Faculty of Science (Series A), Kyusyu (University)	Japan
Mém. Soc. Sci. Liége	Mémoires de la Société Royale des Sciences de Liége	Belgium
Metabolism	Metabolism	U.S.A.
Met. Mag., Lond.	Meteorological Magazine	Great Britain
Metrika	Metrika	Germany
Metroeconomica	Metroeconomica	Italy
Metron	Metron	Italy
Mitt. Ver. Schweiz. Versich.-Math.	Mitteilungen der Vereinigung Schweizerisher Versicherungs-Mathematiker	Switzerland
Monatsh. Math.	Monatshefte für Mathematik	Austria
Monthly Not. R. Astr. Soc.	Monthly Notices of the Royal Astronomical Society	Great Britain
Nat. Bur. Stand., Statist. Rep. & Publ.	National Bureau of Standards, Washington : Statistical Reports and Publications	U.S.A.
Nature	Nature (London)	Great Britain
Ned. Verzek. Actuar. Bij.	Nederlands Verzekeringsarchief, Actuarieel Bijvoegsel, The Hague	Netherlands
Nieuw Arch. Wisk.	Nieuw Archief voor Wiskunde, Groningen	Netherlands
Nieuw Tijdschr. Wisk.	Nieuw Tijdschrift voor Wiskunde, Groningen	Netherlands

Operat. Res. Quart.	Operational Research Quarterly	Great Britain
Osaka Math. J.	Osaka Mathematical Journal, Osaka (University)	Japan
Öst. Ingen.-arch.	Österreichisches Ingenieurarchiv	Austria
Oxf. Econ. Papers	Oxford Economic Papers	Great Britain
Pakistan Econ. J.	Pakistan Economic Journal	Pakistan
Phil. Mag.	Philosophical Magazine	Great Britain
Phil. Phenom. Res.	Philosophical and Phenomenological Research	Great Britain
Phil. Trans.	Philosophical Transactions of the Royal Society	Great Britain
Population Stud.	Population Studies	Great Britain
Portug. Math.	Portugaliae Mathematica	Portugal
Poultry Sci.	Poultry Science	U.S.A.
Proc. Amer. Soc. Hort. Sci.	Proceedings, American Society for Horticultural Science	U.S.A.
Proc. Amer. Soc. Qual. Contr.	Proceedings of the American Society for Quality Control	U.S.A.
Proc. Amer. Soc. Test. Mat.	Proceedings of the American Society for Testing Materials	U.S.A.
Proc. Camb. Phil. Soc.	Proceedings of the Cambridge Philosophical Society	Great Britain
Proc. Indian Acad. Sci.	Proceedings of the Indian Academy of Sciences	India
Proc. Indian Ass. Cult. Sci.	Proceedings of the Indian Association for Cultivation of Science	India
Proc. Indian Sci. Congr.	Proceedings of the Indian Science Congress	India
Proc. Inst. Statist. Math.	Proceedings of the Institute of Statistical Mathematics	Japan
Proc. Japan Acad.	Proceedings of the Japan Academy	Japan
Proc. Kon. Ned. Akad., Wetensch. A	Proceedings, Koninklijke Nederlandsche Akademie van Wetenschappen (A)	Netherlands
Proc. Lahore Phil. Soc.	Proceedings of the Lahore Philosophical Society	India
Proc. Lond. Math. Soc.	Proceedings of the London Mathematical Society	Great Britain
Proc. Math. Phys. Soc. Egypt	Proceedings of the Mathematical and Physical Society of Egypt	Egypt
Proc. Nat. Acad. Sci. India	Proceedings of the National Academy of Sciences	India
Proc. Nat. Inst. Sci. India (A & B)	Proceedings of the National Institute of Sciences of India (Section A & B)	India
Proc. Pakistan Statist. Ass.	Proceedings of the Pakistan Statistical Association	Pakistan
Proc. Qual. Contr. Ass., Bangalore	Proceedings of the Quality Control Association, Bangalore	India
Proc. R. Irish Acad.	Proceedings of the Royal Irish Academy	Ireland
Proc. Roy. Soc. (A & B)	Proceedings of the Royal Society (Series A & B)	Great Britain
Proc. Roy. Soc. N.S.W.	Proceedings of the Royal Society of N.S. Wales	Australia
Proc. Roy. Soc. N.Z.	Proceedings of the Royal Society of New Zealand	New Zealand
Proc. Soc. Exp. Biol., N.Y.	Proceedings of the Society of Experimental Biology and Medicine	U.S.A.
Proc. Soil Sci. Soc. Amer.	Proceedings, Soil Science Society of America	U.S.A.
Przegl. Statyst.	Przeglad Statystyczny	Poland
Psychometrika	Psychometrika	U.S.A.
Public Health Rep.	Public Health Reports	U.S.A.
Public Opinion Quart.	Public Opinion Quarterly	U.S.A.
Publ. Inst. Mat. Estadíst. Fac. Ingen. Agric., Montevideo	Publicaciones del Instituto de Matemáticas y Estadística de la Facultad de Ingenieria y Agrimensura, Montevideo	Uruguay
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Publ. Math., Debrecen	Publicationes Mathematicae	Hungary
Publ. Math. Inst. Hung. Acad. Sci.	Publications of the Mathematical Institute of the Hungarian Academy of Sciences	Hungary
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Quart. J. Exp. Physiol.	Quarterly Journal of Experimental Physiology	U.S.A.
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Quart. J. Med.	Quarterly Journal of Medicine	Great Britain
Quart. J. R. Met. Soc.	Quarterly Journal of the Royal Meteorological Society	Great Britain

RAND Corp., Publ. & Rep.	RAND Corporation, Publications and Reports	U.S.A.
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Rep. Statist. Appl. Res. (JUSE)	Reports of Statistical Application Research, Union of Japanese Scientists & Engineers	Japan
Research, Beirut	Research, A.U.B.—Beirut	Lebanon
Rev. Acad. Madrid	Revista de la Real Academia de Ciencias Exactas	Spain
Rev. Brasil. Estatist.	Revista Brasileira de Estatística	Brazil
Rev. Cienc. Tucumán	Revista de Ciencias de la Universidad Nacional del Tucumán	Argentine
Rev. Écon.	Revue Économique	France
Rev. Econ. Statist.	Review of Economics and Statistics	U.S.A.
Rev. Econ. Stud.	Review of Economic Studies	Great Britain
Rev. Economia, Lisbon	Revista de Economia	Portugal
Rev. Estocást. Soc. Argent. Estadíst.	Revista Estocástica de la Sociedad Argentina de Estadística	Argentine
Rev. Fac. Cienc. Econ. Admin. Montevideo	Revista de la Facultad de Ciencias Economicas y de Administracion, Montevideo	Uruguay
Rev. Int. Statist. Inst.	Review, International Statistical Institute	Netherlands
Rev. Ital. Econ. Demogr. Statist.	Revista Italiana di Economia Demografia e Statistica	Italy
Rev. Mat. Hisp.-Amer.	Revista Matematica Hispano-Americana	Spain
Rev. Rech. Operat.	Revue de Recherche Operationelle	France
Rev. Sci., Paris	Revue Scientifique	France
Rev. Soc. Cubana Cienc. Fis. Mat.	Revista de la Sociedad Cubana de Ciencas, Fisicas y Matemàticas	Cuba
Rev. Statist. Appl.	Revue de Statistique Appliquée	France
Ric. Sci.	Ricerca Scientifica	Italy
Riv. Int. Sci. Soc.	Rivista Internazionale di Science Sociali	Italy
Riv. Polit. Econ.	Rivista di Politica Economica	Italy
Sankhyā	Sankhyā ; Indian Journal of Statistics	India
Schmollers Jb.	Schmollers Jahrbuch	Germany
Schr. Ver. Sozialpolit.	Schriften des Vereins für Sozialpolitik	Germany
Schweiz. Zeit. Volkwirtsch. Statist.	Schweizerische Zeitschrift für Volkwirtschaft und Statistik	Switzerland
Science	Science	U.S.A.
Science & Culture	Science and Culture	India
Scientia	Scientia	Italy
Scottish J. Polit. Econ.	Scottish Journal of Political Economy	Great Britain
Simon Stevin	Simon Stevin, Groningen	Netherlands
Skand. Aktuartidskr.	Skandinavisk Aktuarietidskrift	Sweden
Sociometry	Sociometry	U.S.A.
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South Africa J. Econ.	South African Journal of Economics	South Africa
Statistica, Bologna	Statistica	Italy
Statist. Nachr.	Statistische Nachrichten	Austria
Statist. Neerlandica	Statistica Neerlandica, The Hague	Netherlands
Statist. Praxis	Statistische Praxis	Germany
Statist. Qual. Contr.	Statistical Quality Control	Japan
Statist. Rev. Belgrade	Statisticka Revija	Yugoslavia
Studi. Mercato	Studi di Mercato	Italy
Studia Math.	Studia Mathematica	Poland
Sugaku	Sugaku (Mathematical Society of Japan)	Japan
Tech. News, Nat. Bur. Stand.	Technical News Letter, National Bureau of Standards, Washington	U.S.A.
Tech. Publ., Amer. Soc. Test. Mat.	Technical Publications, American Society for Testing Materials	U.S.A.
Tech. Rep., C.S.I.R.O., Aust.	Technical Reports, Commonwealth Scientific & Industrial Research Organisation	Australia

Technometrics	Technometrics	U.S.A.
Teor. Veroyat. Primen.	Teoriia Veroyatnostey i ee Primenenia	U.S.S.R.
Textile Res.	Textile Research	U.S.A.
Tobacco Sci.	Tobacco Science	U.S.A.
Trab. Estadíst.	Trabajos de Estadística	Spain
Trans. Edin. Math. Soc.	Transactions of the Edinburgh Mathematical Society	Great Britain
Trans. Fac. Actuar., Edinb.	Transactions of the Faculty of Actuaries	Great Britain
Trans. Manchester Statist. Soc.	Transactions of the Manchester Statistical Society	Great Britain
Trans. Nat. Inst. Sci. India	Transactions of the National Institute of Sciences of India	India
Trans. N.Y. Acad. Sci.	Transactions of the New York Academy of Science	U.S.A.
Trans. Roy. Soc. N.Z.	Transactions of the Royal Society, N. Zealand	New Zealand
Trans. Soc. Actuar., Chicago	Transactions of the Society of Actuaries	U.S.A.
Ukrain. Mat. Z.	Ukrainskij Matematiceskij Zurnal	U.S.S.R.
U.N. (F.A.O.) Tech. Publ.	United Nations (Food and Agricultural Organisation) Technical Publications	Italy
U.N. Statist. Publ.	United Nations : Statistical Office Publications	U.S.A.
Unternehmensforschung (O.R.)	Unternehmensforschung (Operations Research)	Germany
U.S. Census Publ.	U.S. Bureau of the Census, Publications	U.S.A.
U.S. Dept. Agric., Publ.	U.S. Department of Agriculture, Publications	U.S.A.
Verh. Kon. Belg. Akad., II	Verhandelingen. Koninklijke Belgische Akademie (Serie 2)	Belgium
Verh. Vlaamse Akad. Belg.	Verhandelingen. Koninklijke Vlaamsche Akademie	Belgium
Vitamins & Hormones	Vitamins and Hormones	U.S.A.
Weltwirts. Arch.	Weltwirtschaftliches Archiv	Germany
Wirtsch. und Statist.	Wirtschaft und Statistik	Germany
Yorkshire Bull. Econ. Soc. Res.	Yorkshire Bulletin of Economic and Social Research	Great Britain
Zastosowania Mat.	Zastosowania Matematyki	Poland
Zeit. angew. Math. Mech.	Zeitschrift für angewandte Mathematik und Mechanik	Germany
Zeit. ges. Staatswiss.	Zeitschrift für die gesamte Staatswissenschaft	Germany
Zeit. Handelswiss. Forsch.	Zeitschrift für Handelswissenschaftliche Forschung	Germany
Zeit. Nat.-Ökon.	Zeitschrift für Nationalökonomie	Austria

The short titles given above are based upon the principles stated in the League of Nations report *International Code of Abbreviations for Titles of Periodicals* (1930) and its *Supplement* (1932) as also used in the *World List of Scientific Periodicals*

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LEVINE, J. (University of North Carolina, Raleigh) **0.4** (11.1)
Monomial-monomial symmetric function tables—*In English*
Biometrika (1959) **46**, 205-213 (4 refs., 8 tables)

These tables give the coefficients in the expression of a product of powers of monomial symmetric functions (not augmented) as a linear sum of non-augmented monomial symmetric functions, for weights up to and including eight. The elements in the products are given " dictionary order " and the arguments of the tables are written down in " dictionary order ". For example, the entry for $(31^3)(2)$ is followed by that for $(31^3)(1^2)$ and then by that for $(31^2)(21)$. A typical item from the table is

$$(2^2)(21^2) = (432)+(421^2)+(3^22)+2(32^21)+3(2^31^2).$$

The tables are preceded by a page of introductory matter which says that they were constructed by using McMahon's D-operator. The tables for higher orders were built up from those of lower order, and were checked by multiplication of factors in different orders. An example is given of a combinatorial application of the tables : namely, that they give directly the evaluation of the number of distinguishable ways of putting a set of objects of different kinds, with given numbers of each kind, into distinguishable boxes so that each box contains a specified number of each kind. Thus, using the entry quoted, there are three ways in which eight objects—two each of three kinds and one each of two futher kinds—may be put into two boxes so that one box contains four objects, two each of two kinds, and the other box contains the rest.

(D. E. Barton)

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MADANSKY, A. (RAND Corporation) **0.5** (6.6)
The fitting of straight lines when both variables are subject to error—*In English*
J. Amer. Statist. Ass. (1959) **54**, 173-203 (53 refs., 2 tables)

This is primarily an expository paper, summarising and relating the work done by quite a number of other investigators.

The problem considered is that of obtaining consistent estimators of the unknown parameters α and β in the relationship $Y = \alpha+\beta X$ when we cannot observe X and Y themselves, but only $x = X+u$ and $y = Y=v$, where u and v represent random errors. Various assumptions are, of course, possible concerning the random variables u, v, and X.

The author considers first estimation by least squares and maximum likelihood. A reasonable least-squares procedure requires knowledge either of the ratio of the variances of u and v or of one of these variances. The method of maximum likelihood requires assumptions concerning the form of the distribution of X, u, and v. If these are assumed to be normal, then it is impossible to estimate α and β uniquely without additional information. The parameters are in this case said to be unidentifiable.

Having shown how additional information may be utilised to enable the estimation of α and β in the unidentifiable case, he deals with information concerning auxiliary or " instrumental " variables. As part of this discussion he deals with the method of grouping as it has been developed from the paper of Wald in 1940.

Finally, the author deals with various estimators for the identifiable case when the distributions of X and Y are unknown, so that the method of maximum likelihood cannot be applied. The methods considered are those of Geary, Neyman, and Wolfowitz. The paper closes with a numerical example, which furnishes an opportunity to compare various estimators and their standard deviations.

(M. R. Dorff)

PFANZAGL, J. (Institute of Statistics, Univ. of Vienna) **0.0 (11.0)**

The axiomatic foundation of a general theory of measurement—*In German*

Schr. des Statist. Inst. der Universität Wien, Neue Folge Nr. 1 (1959) 63 pp. (40 refs.)

By means of the axiomatic method the author attempts to establish a general theory of measurement. The essence of measurement can be reduced to the following general pattern: Given an abstract set M (interpreted as the set of the intensities of the quality to be measured), this set is to be mapped into the set of real numbers. The structural properties inherent to the set M have to be mapped into the set of real numbers in such a way that as much information as possible can be obtained from the structure of the set of real numbers about the structure of the set M. The " structure " of a set M can be interpreted as the relations between elements or pairs of elements of M. The author has set himself the task to establish and examine systems of axioms for various structures of M.

A simple structure which practically occurs in every kind of measurement consists in the order relation between the elements of M, which is defined by a system of five axioms. Together with a topological axiom requiring the connection of M, one thus obtains the structure of a set which by means of a " similar " and " continuous " mapping, turning out to be unique up to monotonous and continuous transformations, can be mapped into the set of real numbers. This mapping is called a " topological " scale.

The theory becomes interesting for practical purposes by the introduction of further structures. For the construction of a " metric " scale the concept of " distance " is relevant. This concept is formally defined by four distance axioms. A metric scale has to reflect the magnitude of these distances in M (the most simple example is probably the location of points on a straight line), and the difference of two real numbers obviously must correspond to the distance in M. Instead of the concept of distance, it is also possible to consider operations between elements of M as a basis for the construction of metric scales. The most familiar example seems to be the addive operation which finds its most evident application in the measurement of length. The paper is focused upon the introduction of a " metric operation " which can be interpreted as a rule of bisection in the special case, when also the commutativity and reflexivity of this operation is required. The essential properties of the metric operation are monotony, continuity and bisymmetry.

After the introduction of the two concepts of " distance " and of " metric operation ", it is shown that the axioms of distance and of metric operation are equivalent in relation to the construction of a metric scale. Such a scale is unique up to linear transformations.

The paper also gives illustrative examples for the application of the general theory to different fields.

(F. Ferschl)

Volume 1. No. 1. CORRECTIONS

ABSTRACT

6. Line 10—read as " Girshick."
11. See also abstract no. 151.
12. The original paper was in English: in journal title, for " Ciendias " read " Ciencias " and the abstractor's name is " Oliveira ".
13. Line 6—delete " (2) " at end of line.
40. Line 14—read " . . . effect of non-normality is not marked ".
48. See also abstract no. 55.
55. See also abstract no. 48.
59. Initials of author should read " J. C."
68. Last line: for " Alk." read " Kut."
69. Title of paper and line 2—read name as " Kolmogoroff ".
73. The Ain Shams University is at Abbasiyah, Cairo.
119. Lines 6 and 7—read name as " Khintchin ".
122. Classification should read " **10.9** (–.–) ".
136. Line 12—read the name as " Darmois ".
151. See also abstract no. 11.

BARTON, D. E. & DAVID, FLORENCE N. (University College, London) **1.3 (2.5)**

Sequential occupancy—*In English*

Biometrika (1959) **46**, 218-223 (5 refs.)

The problem is considered where balls are repeatedly thrown, according to some probability law, into N boxes until just k are left empty. The distribution of the number n of throws needed to achieve this is investigated. The first law considered is the classical one where, the throws being independent, each box is equally likely to receive any ball. The probability distribution function and cumulants of n are derived and two limiting forms of probability distribution function are obtained, namely, the log chi-squared limit as N tends to infinity with k fixed, and the normal limit when both N and k tend to infinity while keeping a constant ratio.

The second law is the Romanovsky law where each box has a maximum content of r balls but balls fall randomly into non-full boxes with proportionally reduced probabilities of falling into partly filled boxes. The probability distribution function of n is derived and its advancing factorial moments. The third law is the generalised Polya or " apparent contagion " law where balls are more likely to fall into boxes which have already received balls. The probability distribution function and advancing factorial moments of n are obtained. The fourth law is where balls are still independent but have different, fixed chances of falling into the various boxes. General expressions are derived for the moments and probability distribution function of the number of empty boxes for the case of a fixed sample number and for the number of balls needed to achieve just k boxes left empty.

Two applications of the latter results are given, firstly when the probabilities of falling into the boxes are obtained by randomly breaking a line of unit length into N parts, and secondly where these probabilities are in arithmetic progression. In the second application the expected value of n is shown to be $O(N^2)$ as compared with $O(N \log N)$ in the classical case.

The probability distribution functions of n are derived by a direct combinatorial method; its moments are derived by considering it as the sum of the numbers of balls needed to reduce the number of empty boxes from N to $N-1$, $N-1$ to $N-2$, etc.

(D. E. Barton)

BÖGE, W. (Mathematisches Seminar, Hamburg) **1.0 (1.5)**

About the characterisation of infinitely divisible probability distributions—*In German*

J. reine angew. Math. (1959) **201**, 150-156 (2 refs.)

In this paper results, obtained in 1954 by N. N. Worobjow for certain probability distributions on finite Abelian groups, are generalised to arbitrary finite groups. The author points out that his results depart in one point (uniqueness of the representation of the logarithm of the characteristic function) from the results of Worobjow.

G may be a finite, multiplicative group which is not assumed to be commutative and b may be a probability distribution on G, then we consider b as an element of the group ring R belonging to G, i.e. R is the set of vectors with the elements of the group as a basis; in the representation of b the respectively corresponding probabilities of the particular elements of the group appear as coefficients. The subset Z of the probability distributions b in R is characterised by the following properties: all coefficients $\geqq 0$ and the sum of the coefficients $= 1$. Here the product of two independent random variables has a probability distribution which is represented by the product (in the group ring R) of the two corresponding elements of R.

In an introductory paragraph on the group ring R, a norm (the sum of the absolute values of the coefficients) is defined and it is shown that the exponential series converges for all elements of R having a norm smaller than 1. The addition theorem holds true and the well known connection is valid between the functions *exp* and *log* defined by the series.

A probability distribution b of Z is called infinitely divisible if for every natural number k there exists a probability distribution $b(k)$ in Z so that b is the power of the order k of $b(k)$. It is the purpose of the present paper to characterise in Z the subset B, which consists of all infinitely divisible probability distributions.

It is proved that, for every probability distribution b of B, there exists a partial sequence of the corresponding $b(k)$ which converges to an idempotent of R and every $b(k)$ is a regular element of the subring belonging to this idempotent. Every b of B can be written in the form $b = exp\ (v)$, in which v belongs to a certain subset, defined by linear homogeneous inequalities, of the above mentioned subring: conversely $exp\ (v)$ belongs to B for every v of this subset.

Using this characterisation of B he proves, in the following paragraph that B is multiplicatively closed if and only if G is an Abelian group. Finally it is shown that the exponential representation of B is unique if and only if all elements of G have the order 2.

(W. Uhlmann)

DERMAN, C. & SACKS, J. (Columbia University) **1.7** (1.6)
On Dvoretzky's stochastic approximation theorem—*In English*
Ann. Math. Statist (1959) **30**, 601-606 (7 refs.)

A simpler proof than those previously presented by Dvoretzky and Wolfowitz is given on the convergence, with probability one and in mean square, of stochastic approximation procedures. Also, the probability one version is extended directly to the multi-dimensional case with absolute values of real numbers replaced by lengths of vectors.

(R. L. Anderson)

DEVINATZ, A. (Washington University) **1.2** (–.–)
On a theorem of Lévy-Raikov—*In English*
Ann. Math. Statist. (1959) **30**, 583-586 (5 refs.)

This note gives a sharpened version of this theorem concerning the product of two characteristic functions.

(R. L. Anderson)

EISENBERG, E. & GALE, D. (Rand Corp. and Brown Univ.) **1.8** (0.8)
Consensus of subjective probabilities: The pari-mutuel method—*In English*
Ann. Math. Statist. (1959) **30**, 165-168

The i-th member of a group of m persons assigns a subjective probability p_{ij} that the j-th of a set of n events will occur, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$. The question is how these m sets of assignments shall be weighted to yield a consensus of subjective probabilities for the group. The pari-mutuel system of betting on horse races is suggested as a possible method. Each person is considered a bettor with a budget b_i to be bet on n horses H_j. He places his money so as to maximise his expectation of winning; that is, he bets amounts $\beta_{ij} > 0$ on those horses H_j for which the ratio p_{ij}/π_j is a maximum, where π_j is the track probability ("odds") on H_j. These track probabilities in turn depend on the bets made: $\sum_{i=1}^{m} \beta_{ij} = \pi_j$. Since each man bets his entire budget, $\sum_{j=1}^{n} \beta_{ij} = b_i$. The unit of money is so normalised that $\sum_{i=1}^{m} b_i = 1$. The strategy of each bettor is expressed as follows: if $\mu_i = \max_s (p_{is}/\pi_s)$, and if $\beta_{ij} > 0$, then $\mu_i = p_{ij}/\pi_j$. The i-th man bets no money on horses H_k for which $p_{ik}/\pi_k < \mu_i$.

It is shown that for any set of p_{ij}'s and b_i's, with $p_{ij} \neq 0$ for some i, all j, there exists a unique set of equilibrium probabilities (track odds) $\pi_j (1 \leqq j \leqq n)$ satisfying the above conditions. These π_j are offered as the consensus of the m sets of subjective probabilities p_{ij}. The set of probabilities π_j and of bets β_{ij} are found by maximising the function

$$\phi(\xi_{11}, \ldots, \xi_{mn}) = \sum_{i=1}^{m} b_i \log \Big(\sum_{j=1}^{n} p_{ij} \xi_{ij} \Big)$$

of mn variables ξ_{ij}, under the conditions $\xi_{ij} \geqq 0$ for all i, j and $\sum_{i=1}^{m} \xi_{ij} = 1$ for all j. Then

$$\pi_j = \max_s [b_i p_{ij} / \Sigma p_{is} \xi_{is}], \text{ and } \beta_{ij} = \xi_{ij} \pi_j,$$

where the ξ_{ij} are the values for which ϕ is a maximum.

It is noted that if two bettors with equal budgets bet on two horses H_1, H_2, and if the first bettor assigns $p_{11} = p_{12} = \frac{1}{2}$, the pari-mutuel method gives equilibrium probabilities $\pi_1 = \pi_2 = \frac{1}{2}$, no matter what the second bettor's subjective probabilities are.

(C. W. Helstrom)

ERDŐS, P. & RÉNYI, A. (Math. Inst., Hungarian Academy of Sciences, Budapest) **1.5** (1.1)
Some further statistical properties of the digits in Cantor's series—*In English*
Acta Math. Acad. Sci. Hung. (1959) **10**, 21-29 (5 refs.)

The authors consider the development of the real numbers of the interval $[0, 1]$ into Cantor's series $x = \sum_{n=1}^{\infty} \mathscr{E}_n(x)/q, q, \ldots, q_n$ where the q_n's are positive integers with $q_n \geqq 2$. Let $f_n(k, x)$ be the number of the digits $\mathscr{E}_1(x), \ldots, \mathscr{E}_n(x)$ which are equal to k and $M_n(x) = \max_k f_n(k, x)$. Let further Q_n denote the sum of the reciprocal values of the numbers $q_1, \ldots, q_n$. In a previous paper by A. Rényi [*Mat. Lapok* (1956) **7**, 77-100] the classical theorem of Borel concerning the relative frequencies of the digits of the decimal expansion of a real number is generalised for the digits in the Cantor's series. Now the behaviour of $M_n(x)$, i.e. the most frequent number among the first n digits is analysed under different conditions. The following theorems are proved:

1. If $\lim_{n\to\infty} Q_n/\log n = +\infty$ then, for almost every x, $\lim_{n\to\infty} M_n(x)/Q_n = 1$.

2. If there are positive constants c_1, c_2 such that every n $c_1 \leqq q_n/n \leqq c_2$ and the limit $\lim_{n\to\infty} Q_n/\log n = \alpha > 0$ exists then, for almost every x, $\lim_{n\to\infty} M_n(x)/Q_n = y(\alpha)$ where $y = y(\alpha) > 1$ and is the unique (real) solution of the equation $y \log y = 1/\alpha$.

3. If $\lim_{n\to\infty} q_n/n = +\infty$ and at the same time $\lim_{n\to\infty} Q_n = +\infty$ then, for almost every x, $\lim_{n\to\infty} M_n(x)/Q_n = +\infty$.

(A. Prékopa)

ERDŐS, P. & RÉNYI, A. (Math. Inst., Hungarian Academy of Sciences, Budapest) 1.5 (4.9)
On the central limit theorem for samples from a finite population—*In English*
Magy. Tud. Akad. Mat. Kut. Intézet. Közleményei (1959) **4**, 49-61 (4 refs.)

This paper deals with the following problem: we choose without replacement a random sample $\{a_{ni_1}, a_{ni_2}, \ldots, a_{ni_{s_n}}\}$ of size s_n from the finite population $\{a_{n1}, a_{n2}, \ldots, a_{nn}\}$ and ask under which conditions will the mean $\sum_{j=1}^{s_n} a_{ni_j}/s_n$ of the sample be distributed asymptotically normally. The following condition (of Lindeberg's type) is proved to be sufficient:

$$\lim_{n\to\infty}\left[\sum_{|a_{nk}|>\epsilon D_{ns_n}} a_{nk}^2 \Big/ \sum_{k=1}^{n} a_{nk}^2\right] = 0$$

for any ϵ, where $\sum_{k=1}^{n} a_{nk} = 0$, $s_n < n/2$ and

$$D_{ns}^2 = s/n\,(1-s/n)\sum_{k=1}^{n} a_{nk}^2.$$

This implies the sufficiency of the following condition of Liapunoff's type too:

$$\lim_{n\to+\infty} (n/s_n)^{\frac{1}{2}}\left(\sum_{k=1}^{n}|a_{nk}|^3\right)\left(\sum_{k=1}^{n} a_{nk}^2\right)^{-3/2} = 0$$

with $\sum_{k=1}^{n} a_{nk} = 0$, $s_n = n/2$.

This result of the authors contains as a special case the previous result of W. G. Madow [*Ann. Math. Statist.* (1948) **19**, 535-545], as well as the well-known extension of the de Moivre-Laplace theorem for the hypergeometric distribution. The proof is carried out by the aid of the characteristic function.

Three examples are given:

1. If $a_{nk} = a_k$ does not depend on n, the condition is satisfied if $s_n \to +\infty$ arbitrary slowly and $0 < c \leqq |a_k| \leqq C$.

2. If $a_{nk} = (-1)^k/\sqrt{k}$ then the condition is satisfied if $\lim_{n\to+8} (\log s_n/\log n) = 1$, e.g. for $s_n \sim n/(\log n)^A$ with arbitrary large $A > 0$.

3. It is impossible to satisfy the condition if $\sum_{k=1}^{n} a_{nk}^2 \leqq \alpha$ and $\max_k |a_{nk}| \geqq \beta > 0$.

(K. Sarkadi)

FABIAN, V. (Res. Inst. of Inorg. Chemistry, Ústí) 1.8 (4.4)
Some modifications of interval estimation and choice of number of observations for a binomial random variable—*In Czech*
Aplik. Mat. (1959) **4**, 35-52 (9 refs.)

The paper deals with decision procedures appropriate to the situation when we do not need the whole amount of information provided by an interval estimate of unknown parameter. In such cases we can, by a slight modification of the usual procedures of interval estimation, achieve decreased probability of a wrong decision without loss of that information in which we are interested.

When we have, for instance, to decide whether the unknown parameter P is greater or less than a given value s and an interval of indifference around s with end points q and r is given, a statistical method choosing between the decisions P greater than q and P less than r will satisfy the needs. Such a method can be based on estimate by an interval (m, M) around a random variable N with following property: there exists a number K such that m is greater than q when N is greater or equal to K and M is less than r, when N is less than K. The decisions P greater than q and P less than r are made according to the position of N with respect to K. The author shows, that if the size of the confidence interval (m, M) is greater or equal to $1-2a$, then the probability of wrong decision is not only at most $2a$, as it is immediately seen, but at most a.

In the second situation studied by the author the choice is between three decisions: (1) P greater than a given value s, (2) P less than s, (3) P in (u, v), when (u, v) is a given interval around s. A suitable method can be constructed, when we have an interval estimate (m, M) with the property, that (m, M) is contained in (u, v) whenever (m, M) contains s. In this case we accept the third decision. In the remaining cases one of the first two decisions is accepted according to the position of (m, M) with respect to s. It is shown, that when the first decision is weakened from P greater than s to P greater or equal to s, the probability of wrong decision is at most a, when the interval (m, M) has the size $1-2a$.

The author deals in detail with the case where P represents the unknown probability of an event A and gives several tables. Here (m, M) is the usual interval estimate of parameter P of binomial distribution, N the relative frequency of A. For the first method (given q, r and a) the smallest sample size n is tabulated, for which the confidence interval (m, M) has for some K the property desired for the use of the method. In the second problem, the author presents tables, giving for prescribed sample size n, level a and parameter-value s, the smallest intervals (c, C) and (l, L) with following properties: if the absolute frequency F of A is less than c, M is less than s; if F is greater than S, m is greater than s; if F is contained in (c, C), (M, m) is contained in (l, L). A decision procedure can be constructed from the tables in analogous way to that we have already described.

To explore the possibilities of extending the table by approximate values, the author compares true values with the normal and inverse-sine approximations.

(P. Mandl)

FINCH, P. D. (London School of Economics) 1.3 (–.–)

Notes on a combinatorial problem of B. O. Koopman—*In English*
Operat. Res. Quart. (1958) **9**, 169-173

The situation is considered where n installations are each connected to $2k$ out of $2n$ stations so that each station is connected to k installations, the detailed scheme of connections being a fixed arrangement A. Some of the stations are then removed, each having an independent probability p of removal, and after this each station services at most one installation—the servicing being arranged so as to maximise the number of installations serviced. The mean value of this is denoted $nI(A)$.

If an installation loses all its stations it cannot be serviced and since this has possibility p^{2k} the mean number of such is np^{2k} whence $I(A)$ is not greater than $1-p^{2k}$. This improves a result of Koopman in the case $k=2$, $p=\frac{1}{2}$. The inequality is strengthened by further eliminating one of each pair of installations with one station left connected to both. This subtracts a further $2p^6(1-p)$ from the upper bound to $I(A)$. By further eliminating one from each of all trios of installations with just two stations left between them the upper bound is reduced by $6p^8(1-p)m/n$, where m is the number of installations sharing not more one station with any other. A lower limit to *the* upper bound to $I(A)$ (as maximised over all arrangements) is found, for $k=2$, by giving the exact value for the arrangement where the installations are connected necklace-wise through pairs of stations.

(D. E. Barton)

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ISBELL, J. R. (University of Washington) 1.8 (–.–)

On a problem of Robbins—*In English*
Ann. Math. Statist. (1959) **30**, 606-610 (2 refs.)

Given two coins with unknown probabilities, p_1, p_2, of coming up heads, the problem is to prescribe a rule for making an infinite sequence of tosses, choosing the coin for the nth toss as a function of the history of the sequence since the $(n-r)$th toss (inclusive). The memory length r is fixed. The aim is to maximise the frequency of heads. The rule proposed by the author is the following: change coins when one coin shows tails r successive times, or when $r-1$ tails with one coin are followed by a tail with the other coin. It was not possible to find a uniformly best procedure, but the above procedure is shown to be uniformly better than one presented by Robbins [*Proc. Nat. Acad. Sci.* (1956) **42**, 920-923] and to be best at least in a special case.

(R. L. Anderson)

KITAGAWA, T. (Kyushu University, Japan) 1.7 (4.7)

Successive processes of statistical controls—*In English*

Mem. Fac. Sci. A, *Kyushu* (1959) **13**, 1-16 (4 refs.)

This is a continuation of a former paper by the same author [*Mem. Fac. Sci.* A, *Kyushu* (1952) **7**, 13-28] and is composed of three parts, No. 3, 4 and 5.

In Part 3, an operation is described of successive shifting applied to some unknown but fixed sequence $\{\eta_i\}$. These η_i's are considered to represent the locations of the object at points in time without the shifting operation. The result u_j of observation made about the location of the object at the time point j under the application of the shifting operation is supposed to contain a random disturbance x_j. These x_j's are supposed to be mutually independent indentically distributed random variables. Given a sequence $\{\xi_i\}$ of reference values and a set $\{c_i, j;\ j=1, 2, \ldots, i;\ i = 1, 2, \ldots\}$ of constants, the shifting operation is performed by successive application of the additional shifting $+\sum_{j=1}^{m} C_{m,j}(\xi_j - u_j)$ to the location of the object after the m-th observation.

In Part 4 the special case of the present controlling process characterised by the quantities

$$\xi_i = \xi;\ \eta = \eta;\ c_{i,i} = c_i;\ c_{i,j} = 0\ (j = i)$$

is compared with the stochastic approximation procedure of Robbins and Monroe. The author emphasises the point that in the present procedure the true values of the locations are unknown to the experimenter while they are taken to be known in the Robbins-Monroe procedure. But it seems that the present procedure is applicable only to the adjustment system of linear type, i.e. the system where desired additional adjustment is possible without the knowledge of the present state of the system. For such a case the difference is only apparent. Theorem 4.1 is a modification of a theorem of Dvoretzky [*Proc. Third Berkeley Symposium* (1956) **1**, 39-55] so as to fit the present procedure where i's may take different values at different i's while they are assumed to be constant in the Dvoretzsky's formulation. Description of the theorem 4.1 is vague and it seems that we must interpret the notations used here according to the definitions of the Dvoretzky's paper. In Theorem 4.2 the controlling of Robbins-Monroe type is discussed by using the result of Theorem 4.1. For this case it is assumed that the ξ_i's must satisfy the condition $\sum_{i=1}^{\infty} (\xi_i + 1 - \xi_i)^2 < +\infty$, which implies that the ξ_i's form a Cauchy sequence. The proof of Theorem 4.2 is incorrect.

In part 5 a continuous time parameter representation of the model is indicated where the Brownian motion process is used to represent the random disturbance inherent in the observation.

(H. Akaike)

LE CAM, L. (A. P. Sloane Foundation) 1.5 (2.6)

A theorem on the division of an interval by the points taken at random—*In French*

Publ. Inst. Statist. Paris (1958) **7**, 7-16 (3 refs.)

This theorem compares the limits of the probability distribution of the following variables: $\sum_{j=1}^{n} g_n(v_j)$, g being a sequence of Baire functions and v a sequence of random variables independently distributed according to the density $1/e^x$ and $\sum_{j=1}^{n} g_n(u_{n,j})$. $1/n(u_{n,j})$ is the length of the j-th interval separating two consecutive points in a group of n points chosen at random on the segment (0 ,1) with a uniform probability distribution.

(D. Dugué)

MAREZEWSKI, B. & STEINHAUS, H. (Math. Inst., Polish Acad. Sciences, Wroclaw) 1.0 (10.9)
On a certain distance of sets and the corresponding distance of functions—*In English*
Colloq. Math. (1958) **6**, 319-327 (2 refs.)

This paper is devoted to a mathematical study of a new distance of sets and the corresponding distance of functions which, in the authors' opinion, are useful in several practical applications, especially in some biological problems.

The enumerably additive measure μ defined on an enumerably additive field of subsets of a given space X is given and the authors define the distance $\sigma(A, B)$ of two sets as a ratio of the measure $\mu(A \dot{-} B)$ of the symmetric difference of A and B to the measure $\mu(A \cup B)$ of their union. Analogously the distance $\sigma(f, g)$ of two μ-integrable functions f and g, defined on X, is defined as a ratio of the μ-integral of the absolute difference $|f-g|$ to the μ-integral of the maximum of three quantities $|f|, |g|, |f-g|$. After showing that $\sigma(A, B)$ has all properties of distance, and that it is bounded by 1, some continuity properties of σ with respect to μ are stated. It is also shown that the resulting metric space is complete. Beyond necessary formal changes, the considerations concerning $\sigma(f, g)$ run parallel. In particular, the equivalence, in a certain sense, of the two introduced notions of distance is stated.

An example of the application of the two distances in biology is described in the third part of the paper. The problem is to characterise numerically the biotopical difference of two forests with respect to the species they include Consider the set A of all species growing in the first forest, and set B of all species growing in the other forest, and the number of elements in a set as the measure μ. It is proposed to take the distance $\sigma(A, B)$ as the characteristic sought for. This distance will assume its greatest value 1 if there are no species common to the two forests, and its smallest value 0, if the two forests are identical as to the species they contain. If we were interested not only which species grow in the forests but also, how often they appear, we would take instead of A and B functions defined in the union of A and B: function f would be the quantitative characteristic of the species growing in the first forest, function g the analogous characteristic of the species growing in the other. The distance $\sigma(f, g)$ is then the quantitative characteristic of the biotopical difference between the two forests considered.

The publication of another paper, (written in Polish and to appear in *Zastosowania Matematyki*) is announced in which the applications will be examined in more detail.

(S. Zubrzycki)

RÉNYI, A. & RÉVÉSZ, P. (Eötvös University, Budapest) 1.5 (10.1)
On mixing sequences of random variables—*In English*
Acta Math. Acad. Sci. Hung. (1958) **9**, 389-393 (10 refs.)

In a previous paper A. Rényi [*Acta Math. Acad. Sci. Hung.* (1958) **9**, 215-228] introduced the notion of a mixing sequence of events. The sequence $C_0 = \Omega$, C_1, C_2, ... $(P\{C_n\}>0)$ is called "mixing" if for every k, and every event B with $P\{B\}>0$, $\lim P\{C_n \mid B\} = \lambda$ $(0<\lambda<1)$. It is proved that if $\lim_{n\to\infty} P\{C_n \mid C_k\} = \lambda$ for every $k = 0, 1, 2, \ldots$, then the sequence C_k is mixing.

In the present paper the authors consider mixing sequences of random variables. The sequence of random variables $\zeta_1, \zeta_2, \ldots$ is called mixing if there is a distribution function $F(x)$ such that for every event B with $P\{B\}>0$ we have $P\{\zeta_n<x \mid B\} \Rightarrow F(x)$. It is proved that the sequence $\zeta_1, \zeta_2, \ldots$ is mixing if and only if for every k there exists a linear Borel set E_k such that $P\{\zeta_k \epsilon E_k\} = 1$ and

$$P\{\zeta_n<x \mid \zeta_k = y\} \Rightarrow F(x)$$

for every $y \epsilon E_k$ where $F(x)$ is a distribution function. Examples are given for sequences of random variables constituting Markoff-chains.

(A. Prékopa)

RÉVÉSZ, P. (Eötvös University, Budapest) 1.5 (1.0)

On the limit distributions of sums of dependent random variables—*In English*

Ann. Univ. Sci. Budapest. (Sect. Math.) (1958) 1, 135-142 (7 refs.)

A. Rényi [*Acta Math. Acad. Sci. Hung.* (1950) 1, 99-100] proved that if $\xi_1, \xi_2, \ldots$ are independent random variables with regard to a probability measure μ, and A_n, B_n are sequences of real numbers, $B_n \to \infty$ as $n \to \infty$ and

$$\mu(\xi_1 + \ldots + \xi_n / B_n - A_n < x) \Rightarrow F(x),$$

then we have also

$$\nu(\xi_1 + \ldots + \xi_n / B_n - A_n < x) \Rightarrow F(x),$$

where ν is an arbitrary measure which is absolutely continuous with respect to x.

The author now examines the inverse problem, i.e. if we have a sequence of (not independent) random variables, $\xi_1, \xi_2, \ldots$, and the probability measure is ν under what conditions can a new measure μ be introduced such that $\nu << \mu$ and the first expression given above holds? In this way conditions can be given for the existence of limiting distributions of sums of random variables which are not independent. Let $\xi_1, \xi_2, \ldots$; $\xi_1^*, \xi_2^*, \ldots$ sequences of random variables, where ξ_i and ξ_i^* are distributed and $\xi_1^*, \xi_2^*, \ldots$ are independent. Suppose that for the ξ_k^*'s a relation of the kind stated first above holds with P instead of μ; and to every $\mathscr{E}$ there is a δ such that if

$$\sum_{i=1}^{r} P(x_1^{(i)} \leqq \xi_i < y_1^{(i)}) \ldots P(x_n^{(i)} \leqq \xi_n < y_n^{(i)}) < \delta,$$

then

$$\sum_{i=1}^{r} P(x_1^{(i)} \leqq \xi_1 < y_1^{(i)}, \ldots, x_n^{(i)} \leqq \xi_n < y_n^{(i)}) < \mathscr{E}$$

for any system of intervals $[x_1^{(i)}, y_1^{(i)}), \ldots, [x_n^{(i)}, y_n^{(i)})$.

In this case we have also

$$P(\xi_1 + \ldots + \xi_n / B_n - A_n < x) \Rightarrow F(x).$$

Another application of the theorem of Kolmogoroff is the following. Let $\nu_1, \nu_2, \ldots$ be a sequence of positive integer-valued random variables. $\nu_1^*, \nu_2^*, \ldots$ is another one consisting of independent variables and having the same distributions as the corresponding former ones. If there is a constant C such that for every n

$$P(\vec{\xi_n} \in K_n, \vec{\nu_n} \in L_n) < C P(\vec{\xi_n} \in K_n) P(\vec{\nu_n} \in L_n)$$

where K_n and L_n are arbitrary n-dimensional intervals and

$$P(\xi_1 + \ldots + \xi_{\nu_n} / B_n - A_n < x) \Rightarrow F(x),$$

then this limit holds also with ν_n^* instead of ν_n.

(A. Prékopa)

RÉVÉSZ, P. (Eötvös University, Budapest) 1.5 (6.9)

A limit distribution theorem for sums of dependent random variables—*In English*

Acta Math. Acad. Sci. Hung. (1959) 10, 125-131 (5 refs.)

In order to prove a limit distribution theorem for sums of not necessarily independent random variables, the author uses a general theorem of Rényi [*Acta Math. Acad. Sci. Hung.* (1958) 9, 215-228]. In that paper, Rényi proved that, if $\xi_1, \xi_2, \ldots$ are independent random variables and $P(\zeta_n - A_n / B_n < x) \Rightarrow F(x)$, where $F(x)$ is a distribution function, $\zeta_n = \xi_1 + \ldots + \xi_n$; A_n, B_n are sequences of real numbers ($\lim_{n \to \infty} B_n = +\infty$), then for every other probability measure Q—which is absolutely continuous relative to P—we have also

$$Q(\zeta_n - A_n / B_n < x) \Rightarrow F(x).$$

The theorem proved by the author is the following: if $\xi_1, \xi_2, \ldots$ is a sequence of random variables, then $\xi_1^*, \xi_2^*, \ldots$ is a sequence of independent random variables with the property that $P(\xi_i < x) = P(\xi_i^* < x)$ $(i = 1, 2, \ldots)$. Moreover, $P(\xi_1^* + \ldots + \xi_n^* / B_n - A_n < x) \Rightarrow F(x)$ where $F(x)$ is a distribution function and A_n, B_n are sequences of real numbers. Then, provided the inequality

$$\begin{aligned} &| P(a_1 \leqq \xi_1 < b_1, \ldots, a_n \leqq \xi_n < b_n) \\ &\quad - P(a_1 \leqq \xi_1 < b_1, \ldots, a_{n-1} \leqq \xi_{n-1} < b_{n-1}) P(a_n \leqq \xi_n < b_n) | \\ &\quad \leqq \delta_n P(a_1 \leqq \xi_1 < b_1, \ldots, a_{n-1} \leqq \xi_{n-1} < b_{n-1}) P(a_n \leqq \xi_n < b_n) \end{aligned}$$

holds for any n and for any system of intervals $[a_i, b_i)$ $(i = 1, 2, \ldots, n)$ where $\sum_{n=1}^{\infty} \delta_n^2 < \infty$, we have also

$$P(\xi_1 + \ldots + \xi_n / B_n - A_n) \Rightarrow F(x).$$

(A. Prékopa)

SCHMID, P. (Swiss Forest Res. Inst. and Federal Inst. of Technology) 1.5 (5.6)

On the Kolmogoroff and Smirnoff limit theorems for discontinuous distribution functions—*In English*

Ann. Math. Statist. (1958) **29**, 1011-1027 (24 refs.)

Given a population cumulative distribution function $F(x)$ and its corresponding empirical function $S_N(x)$, based on a random sample of N, i.e. $S_N(x) = k/N$ if exactly k of the N sample values are less than or equal to x.

The Kolmogoroff and Smirnoff distributions for large N of the largest deviation between $F(x)$ and $S_N(x)$ are extended to discontinuous distribution functions. The same extension is made for Rényi's theorems on proportional deviations. These limit distributions are found to depend on the values of $F(x)$ at the points of discontinuity, but not on the form of the function between these points. These deviation statistics are useful to test if a sample comes from a population with a given distribution function $F(x)$.

The following statistics are considered:

(1) The largest absolute deviation,

$$\sup \mid S_N(x) - F(x) \mid N^{1/2}.$$

(2) The largest deviation, omitting the absolute value restriction of (1).

(3) The largest proportional absolute deviation, with $F(x)$ restricted to values greater than zero,

$$\sup \mid S_N(x) - F(x) \mid N^{1/2}/F(x).$$

(4) The largest proportional deviation.

(F. J. Massey)

SCHNELL, E. (Math. Inst., Hungarian Academy of Sciences, Budapest) 1.5 (–.–)

On a conditional limiting distribution theorem—*In English*

Magy. Tud. Akad. Mat. Kut. Intézet. Közleményei (1959) **4**, 3-10 (4 refs.)

This paper deals with the following problem: under what conditions will the relation

$$\lim_{n\to\infty} P\left\{a_1 \leqq \sum_{i=1}^{n} \xi < b_1 \mid a_2 \leqq \sum_{i=1}^{n} \xi_i < b_2\right\} = b_1 - a_1/b_2 - a_2 \quad (a_2 \leqq a_1 < b_1 < b_2)$$

be valid, where $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ are independent and identically distributed variates. From earlier results it is known that, if the distribution of ξ_i is non-latticed and possess an expectation and finite variance, the expression given above is fulfilled if either the third absolute moment is finite [G. Kallianpur & H. Robbins, *Duke Math. J.* (1954) **21**, 285-387] or ξ_i has a quadratically integrable density function [A. Rényi, *Acta Math. Acad. Sci. Hung.* (1955) **6**, 285-335—especially pp. 315-316]; Kallianpur and Robbins show some further cases where the above expression holds.

In the present paper the following condition is proved to be sufficient: $\limsup_{|t|\to\infty} \mid \phi(t) \mid < 1$ where $\phi(t)$ denotes the characteristic function of ξ_i and ξ_i has O expectation and finite variance. This condition is ensured if, in the canonical decomposition of the distribution function of ξ_i, the absolute continuous component has a positive weight. The proof is based on Laplace's method. In addition, the author shows that the sufficiency of the condition mentioned by Kallianpur and Robbins can be simply derived from an estimation by C. G. Esseen [*Acta Math., Stockh.* (1954) **77**, 1-125]. See also the book by B. V. Gnedenko and A. N. Kolmogoroff: *Limit distribution for sum of independent random variables* (1954), Addison-Wesley, Cambridge, Mass., in particular at p. 213.

(K. Sarkadi)

STEINHAUS, H. & TRYBULA, S. (Math. Inst., Polish Acad. Sciences, Wroclaw) 1.0 (5.4)
On a paradox in applied probabilities—*In English*
Bull. Acad. Sci., III, (1959) **7**, 67-69

Let x, y, z, denote three independent variables. It can happen that each of the inequalities $x<y$, $y<z$, $z<x$ holds with probability greater than $\frac{1}{2}$. This paradox makes difficult comparison of three lots of some product by comparing pairs items belonging to any two of the three lots. Moreover, the calculation of the expected value and the variance does not remove this paradox.

(L. Kubik)

TROTTER, H. F. (Kingston, Canada) 1.5 (2.3)
An elementary proof of the central limit theorem—*In English*
Arch. Math., Karlsruhe (1959) **10**, 226-234 (2 refs.)

This proof of the central limit theorem serves mainly didactic purposes. It is elementary in the sense that it avoids the use of characteristic functions. Instead, with each real random variable x with distribution function V the author associates the operator T_x in the space C of all bounded uniformly continuous real functions on $(-\infty, +\infty)$ defined by $(T_x f)(\eta) = \int f(\eta+\xi)dV(\xi)$. This linear operator is a contraction relative to the norm $||f|| = \sup |f|$, and $T_{x+y} = T_x T_y$ if x and y are independent. The convergence of distributions can easily be deduced from the convergence in the norm of the associated operators on the space of all functions of C whose first and second derivatives exist and belong to C. The latter follows, in the case of the central limit theorem under Lindeberg's condition, from the equation $T_{x+y} = T_x T_y$, the inequality

$$||A_1 \ldots A_n f - B_1 \ldots B_n f|| \leq \Sigma_i ||A_i f - B_i f||$$

for commuting contraction operators in C and Taylor's theorem.

The only fact about the normal distribution used in the argument is that the sum of two independent normal variables with mean 0 and variations σ^2 and τ^2 is normal with mean 0 and variance $\sigma^2+\tau^2$. The generalisation to the multivariate case is briefly indicated.

Remark of the abstractor: the method of this paper appears to be related to the more general "algebraic" method used by H. Bergström [*Abh. Math. Sem. Univ. Hamburg* (1959) **23**, 228-256] and earlier papers quoted there.

(W. Krickeberg)

WESLER, O. (University of Michigan)

A classification problem involving multinomials—*In English*

Ann. Math. Statist. (1959) **30**, 128-133 (5 refs.)

This paper uses the notation and terminology of Blackwell & Girshick's *Theory of Games and Statistical Decisions* and develops a procedure for deciding between two given loadings of a k-faced die, i.e. one loading is given by specifying $p = (p_1, p_2, \ldots, p_k)$, the multinomial probabilities of occurrence of the k sides of a die, and specified alternative loading by $q = (q_1, q_2, \ldots, q_k)$. These loadings are not assigned to particular locations of the die.

The problem is a fixed sample size two-decision statistical problem. The solution is obtained by a game-theoretic minimax method. This paper is a summary without proofs of a Standford University Technical Report. Presented here are the general solution, an approximate solution for large sample size and a discussion of the binomial case ($k = 2$) where the die is referred to as a coin. Here the choice is between the probability of heads being p_1 or $p_2 = 1 - p_1$ or being q_1 or $q_2 = (1 - q_1)$. For $k = 2$ the approximate solution for large N is exact and the procedure (assuming that $1 > p_1 > q_1 \geqq \frac{1}{2}$ and r_1 is the number of heads appearing in N tosses) is the two sided symmetric test of the form: decide for p if $r_1 < j$ or $r_1 > N - j$: decide for q if $r_1 > j$ and $r_1 < N - j$; mix actions if $r_1 = j$ or $r_1 = N - j$.

The relationships to similar problems is indicated. These classification problems involving multinomials concerned with the decision as to whether or not a die is loaded are: 1. A problem assuming a specified loading; 2. a problem in which an unknown loading is estimated from one sample and used in a decision for a second die between agreement with the estimated loadings and one with equal probabilities; 3. the problem of determining which of k faces carries a known extra load.

(W. J. Dixon)

AGGARWAL, O. P. & GUTTMAN, I. (Purdue University and Princeton University) 2.8 (5.1)
Truncation and tests of hypotheses—*In English*
Ann. Math. Statist. (1959) **30**, 230-238 (25 refs., 3 tables)

In this paper, the authors examine the loss of power when using tests based on the assumption that the variable being sampled has a " complete " normal distribution when in fact the distribution is a " truncated " one. The cases considered are for small sample sizes (≤ 4) and " symmetric " truncation, while the hypothesis considered is the one-sided testing for the mean of the normal distribution, with σ^2 and truncation point known.

Using the standard uniformly most powerful test for the one-sided testing problem for the mean of a normal distribution, (referred to as the " usual " test) the authors provide expressions for the power of the test for the " complete " normal and for the " truncated " normal distribution. They provide tables of these expressions for the power, the difference between these expressions (the " loss of power "), and the ratio of this loss to the power of the test for the complete normal distribution. The tables are given for tests of size 0·05 based on samples of sizes 1, 2, 3 and 4 and for truncation points 1·5, 2·0, 2·5 and 3·0 standard deviation units from the mean. They also provide the uniformly most powerful test of size α for the one-sided testing problem for the mean of a symmetrically truncated normal distribution (referred to as the " correct " test) and give tables of the significance points for such tests with sizes 0·10, 0·05, 0·025, 0·01, 0·005 based on samples of sizes 1, 2, 3 and 4 when sampling from normal distributions symmetrically truncated at 1·0, 1·5, 2·0, 2·5 and 3·0 standard deviation units from the mean. In addition, tables are given for the power of this " correct " test, and the gain and percentage gain in power of this test over that of the " usual " test.

From these tables it appears that an appreciable loss occurs only in the size of the usual test with the consequence that applying the usual test is to err on the conservative side in making it much more difficult to reject the true hypothesis. The loss in power is found to decrease very rapidly with the distance of the alternative value of the mean from the one tested and also with the distance of the truncation from the mean. In a future paper, the authors propose to give extensive tables of the mean of samples from truncated distributions and to examine the tests at other than 5 per cent. significance levels.

(J. W. Wilkinson)

BOYD, A. V. (Johannesburg, Union of South Africa) 2.7 (3.6)
The harmonic mean of independent $\Gamma(\frac{1}{2})$ variates—*In English*
Simon Stevin (1959) **33**, 34-37

If the independent random variables x_i ($i = 1, \ldots, n$) have a $\Gamma(\lambda_i; \alpha)$-distribution (that is $2\alpha x_i$ has a χ^2-distribution with $2\lambda_i$ degrees of freedom) then their arithmetic mean has a $\Gamma(\sum_i^n \lambda_i; n\alpha)$-distribution. However, if the variables x_i have different parameters α_i the distribution of their arithmetic mean is no longer of the Γ type. The author proves that if the x_i have a $\Gamma(\frac{1}{2}; \alpha_i)$-distribution their harmonic mean, $n/\sum_i x_i^{-1}$, has a $\Gamma(\frac{1}{2}; (\Sigma\sqrt{\alpha_i})^2/n)$-distribution. This is not true if $\lambda_i \neq \frac{1}{2}$; although for $\lambda_i = \frac{1}{2} + n_i$, with n_i a positive integer, the frequency function of the harmonic mean can be expressed in elementary functions, as is shown by an example.

(G. Klerk-Grobben)

FERRERI, C. (School of Statistics, Univ. of Palermo) 2.5 (1.1)

On some properties and on the application of a new stochastic model of dependence—*In Italian*

Statistica (1959) **19**, 63-71 (11 refs., 1 table)

As in Polya's scheme, the random events are not independent in the stochastic model suggested by the author. The modification, with respect to the classical form, consists of the fact that the stochastic dependence must be exclusively derived from " successes ". Thus, at each drawing, $1+\Delta$ white balls are replaced in the urn if the drawn ball is white; but only one ball is replaced if the drawn ball is black. According to this scheme, the author gives the formula for the special case of rare events and weak dependence. For this particular probability distribution, and by means of moment generating functions, he determines the moments, cumulants and factorial moments.

This paper also deals with the problem of simultaneous estimation of the parameters. The frequency distribution of wars in time (Richardson) is utilised to give a concrete application of the theoretical scheme. The author justifies such an application by the consideration that the probability of a war is not independent of the preceding wars.

(V. Amato)

GLASGOW, M. O. (University of Texas) 2.1 (2.9)

Note on the factorial moments of the distribution of locally maximal elements in a random sample—*In English*

Ann. Math. Statist. (1959) **30**, 586-590 (3 refs.)

The results of Austin, Fagen, Lehrer and Penney [*Ann. Math. Statist.* (1957) **28**, 786-790] are extended to include a general recurrence relation for the factorial moments of the distribution of locally maximal elements in a random sample.

(R. L. Anderson)

KESTEN, H. & MORSE, N. (Cornell University) **2.4** (9.4)

A property of the multinomial distribution—*In English*

Ann. Math. Statist. (1959) **30**, 120-127 (1 ref.)

This paper proves a property of the multinomial distribution which is fundamental to the choice of sample size for the selection procedure described in the paper by R. E. Bechhofer, S. Elmaghraby and N. Morse, entitled "A single-sample multiple-decision procedure for selecting the multinomial event which has the highest probability," [*Ann. Math. Statist.* (1959) **30**, 102-119]. The latter paper (see abstract No. 275) and the paper herein summarised will be referred to below as papers [1] and [2], respectively.

Given an experiment whose outcome is one of k possible events, paper [1] sets up the following procedure for selecting, on the basis of N performances of the experiment, that event which has the highest probability: with no ties for first place in frequency of occurrence, select that event which occurred most often, otherwise, select randomly one of those tied for first place, having arranged that each of these have the same probability of selection. The problem of determining N for a required probability of correctly selecting the most probable event was solved in paper [1]. It is pointed out that the answer depends on the configuration of the probabilities of the k events, these probabilities being denoted by $p_{[1]}, \ldots, p_{[k]}$ when written in ascending order. The configuration is measured by $\theta = p_{[k]}/p_{[k-1]}$. The problem is expressed in terms of finding the smallest N so that, for predetermined θ^* and P^*,

$$\text{Probability}\{\text{correct selection} \mid \theta \geqq \theta^*\} \geqq p^*.$$

The least favourable configuration of the p's is defined in paper [1] to be that which for any given N, k, θ^* minimises the probability of a correct solution when $\theta \geqq \theta^*$.

Paper [1] states, and paper [2] gives the proof, that the least favourable configuration is given by

$$p_{[k]}/p_{[i]} = \theta^* \qquad (i = 1, 2, \ldots, k-1),$$

or, since $\sum_{i=1}^{k} p_{[1]} = 1$, by

$$p_{[i]} = 1/(\theta^* + k - 1) \quad (i = 1, 2, \ldots, k-1)$$
$$p_{[k]} = \theta^*/(\theta^* + k - 1).$$

(R. Hooke)

KLERK-GROBBEN, G. (Mathematical Centre, Amsterdam) **2.7** (2.5)

The Γ-distribution—*In Dutch*

Math. Centre, Statistics Dept., Report S250 (1959)

The general Γ-distribution has the density function $g(x;\ \alpha, \beta, \gamma) = \{\beta^{\gamma}\Gamma(\gamma)\}^{-1}(x-\alpha)^{\gamma-1}\exp\{-\beta^{-1}(x-\alpha)\}$. Special cases are the χ^2-distribution with ν degrees of freedom ($\alpha = 0$; $\beta = 2$; $\gamma = \nu/2$) and the exponential distribution ($\gamma = 1$). This report, which is a survey of the literature on Γ-distributions, mentions such subjects as moments, shape of the density function for different values of γ and the limit distribution for $\gamma \to \infty$. It also deals with the χ^2-tables of Pearson and Hartley (*Biometrika Tables*, 1954), the Pearson (1946) and Salvosa (1930) tables of the Γ-distribution. In this connection the report shows the use of the latter tables for finding significance levels of the Poisson distribution, and the relation between the exponential and the Poisson distributions.

Among the properties of Γ-distributed variables, the most important are: (1) if the x_i are independently $\Gamma(\alpha_i, \beta, \gamma_i)$ distributed then Σx_i has a $\Gamma(\Sigma\alpha_i, \beta, \Sigma\gamma_i)$ distribution; (2) if the x_i are independently $\Gamma(0, \beta, \gamma_i)$ distributed then Σx_i is independent of every function which is homogeneous of degree zero in the x_i, thus x_1+x_2 and $x_1/(x_1+x_2)$ are independent; the latter having a B-distribution with parameters γ_1 and γ_2.

With regard to exponentially distributed variates, Pitman [*Proc. Camb. Phil. Soc.* (1937) **33**] gave the following properties: if the smallest observation in a sample $x_1, \ldots, x_n$ is denoted by x_0, then x_0 has a $\Gamma(\alpha, \beta/n, 1)$-distribution and $\sum_1^n (x_1 - x_0)$ a $\Gamma(0, \beta, n-1)$-distribution. If $T(x_1, \ldots, x_n)$ is such that $T(x_1, \ldots, x_n) = T(x_1+c, \ldots, x_n+c)$ for arbitrary constant c, then x_0 and T are independent, e.g. x_0 and $\Sigma(x_i - x_0)$ are independent. If $G(x_1, \ldots, x_n)$ is a function such that $G(kx_1, \ldots, kx_n) = G(x_1, \ldots, x_n)$ and also $G(x_1+c, \ldots, x_n+c) = G(x_1, \ldots, x_n)$ for arbitrary constants k and c then G, x_0 and $\sum_i^n (x_i - x_0)$ are independent of each other.

The main part of the paper is devoted to estimates, tests and confidence intervals for parameter values based on these properties of Γ-distributed variates. If all parameters are unknown, little can be done except to find estimates of the parameters (from moment estimates or approximations to maximum likelihood estimates). These, however, seem to have little accuracy—as is shown by a sampling experiment. If α and γ are known, there exist unbiased estimates for β and $1/\beta$; a confidence interval for β and a test for $\beta = \beta_0$. If β and γ are known, an estimate and a confidence interval for α and a test for $\alpha = \alpha_0$ can be given. In case $\gamma = 1$, estimates together with confidence intervals and tests of significance can be given in a large number of situations among which are life tests and comparisons of two or more different exponential distributions.

(G. Klerk-Grobben)

LESLIE, D. C. M. (Royal Aircraft Establishment, Farnborough) 2.2 (2.7)

Determination of parameters in the Johnson system of probability distributions—*In English Biometrika* (1959) **46**, 229-231 (2 refs.)

The "S_U" distribution is defined as the distribution of a variable y, when the inverse hyperbolic sine of a linear function of y is normally distributed. It can be shown that, by appropriate choice of parameters, distributions with shape factors β_1, β_2 corresponding to all Pearson Types IV and V, and some Type VI curves, can be obtained.

The mathematical determination of these parameters is sometimes rather troublesome. The author puts forward a fairly simple approximate method of carrying out these calculations, and gives some numerical illustrations to illustrate the accuracy of the method.

(N. L. Johnson)

PEARSON, E. S. (University College, London) 2.2 (2.7)

Note on Mr Quenouille's Edgworth Type A transformation—*In English Biometrika* (1959) **46**, 203-204 (3 refs., 1 table)

In representing a non-normal frequency function by a limited number of terms of an Edgeworth Type A expansion, statisticians have frequently used a form containing κ_3, κ_4, and κ_3^2. The tables of random deviates from Edgeworth Type A distributions given by Quenouille [*Biometrika* (1959) **46**, 178-202] were derived from a normally distributed variable through Cornish-Fisher expansions and these distributions will not correspond exactly to particular cases of standard form quoted in this Note.

The extent of the differences is examined and the conclusion is drawn that, except at the lower tail of the x_5 distribution, it would be difficult to detect any such differences—even in samples as large as 1000. It is pointed out that the cumulant-values used by Quenouille for x_5 and x_7 do not correspond to positive definite frequency functions.

(N. W. Please)

Editorial Note: see also abstract No. 325.

RAMASUBBAN, T. A. (London School of Economics) 2.1 (2.5)

The generalised mean difference of the binomial and Poisson distribution—*In English*
Biometrika (1959) **46**, 223-229 (6 refs., 2 tables)

The paper extends some results given by Ramasubban [*Biometrika* (1958) **45**, 549-556] and is concerned with establishing formulae for the generalised mean difference of order r for the binomial and Poisson distributions.

Mean differences of even order can be expressed quite simply in terms of the population moments. Odd order mean differences require some algebraic manipulation and the author gives combinatorial expressions appropriate to a binomial parent population which may be expressed in terms of the hypergeometric function.

Odd order absolute mean differences for the Poisson distribution may be obtained as a limit to those for the binomial distribution; the resulting expressions involve modified Bessel functions of the first kind.

It is pointed out that whilst the paper contains explicit results, it would seem possible to obtain certain recurrence relations connecting the mean differences of different order.

(J. G. Saw)

Editorial Note: see abstract No. 31.

RÉNYI, A. (Math. Inst., Hungarian Academy of Sciences, Budapest) 2.9 (1.5)

Some remarks on the theory of trees—*In English*
Magy. Tud. Akad. Mat. Kut. Intézet. Közleményei (1959) **4**, 73-85 (8 refs.)

It is known that the number of trees of order n, that is to say, of trees with labelled vertrices $P_1, P_2, \ldots, P_n$ is n^{n-2}. The author investigates the distribution of the number ν_n of end-points of a random tree of order n, i.e. of a tree chosen at random from the n^{n-2} possible trees.

This distribution is given by the formula

$$P\{\nu_n = k\} = n!\,\gamma_{n-2}^{n-k}/k!\,n^{n-2}$$

where γ_N^m denote the Stirling-numbers of the second kind $\gamma_N^m = \sum_{r=1}^{m} (-1)^{m-r} r^N \binom{m}{r}$. For the expected value and variance he obtains

$$M\{\nu_n\} = n(1-1/n)^{n-2}$$

$$D^2\{\nu_n\} = n(n-1)(1-2/n)^{n-2} + n(1-1/n)^{n-2} - n^2(1-1/n)^{2n-4},$$

therefore

$$\lim_{n\to+\infty} [M\{\nu_n\}/n] = 1/e$$

$$\lim_{n\to+\infty} [D^2\{\nu_n\}/n] = (e-2)/e^2.$$

Furthermore it is proved that ν_n is in the limit normally distributed; more exactly that

$$\lim_{n\to\infty} P\left[\{(\nu_n - n/e)/e^{-1}\sqrt{n(e-2)}\} < x\right] = 1/\sqrt{2\pi}\int_{-\infty}^{x} \exp(-u^2/2)\,du$$

This is proved by showing that the characteristic function of the standardised form of ν_n tends to the characteristic function of the normal distribution.

(K. Sarkadi)

KSHIRSAGER, A. M. (University of Bombay)
Bartlett decomposition and Wishart distribution—*In English*
Ann. Math. Statist. (1959) **30**, 239-241 (6 refs.)

3.9 (0.6)

Hitherto, only complicated methods have been available for deriving the decomposition theorem due to Bartlett and the Wishart distribution.

A paper in 1951 by Wijsman [*Ann. Math. Statist.* **28**, 415-422] simplifies the procedure by using orthogonal matrices depending upon certain random vectors. This present note continues to simplify the procedure by using orthogonal vectors: the method uses transformations rather than densities and Jacobians.

(R. L. Anderson)

KUIPER, N. H. (Landbouwhogeschool, Wageningen, Netherlands)
Alternative proof of a theorem of Burnbaum and Pyke—*In English*
Ann. Math. Statist. (1959) **30**, 251-252 (2 refs.)

3.8 (2.6)

In 1958, Birnbaum and Pyke [*Ann. Math. Statist.* **29**, 179-182] stated a theorem (No. 3) concerning a distribution related to the maximum distance between empirical and expected distribution functions. An alternative proof that the distribution in question was uniform (0, 1) was given by Dwass [*Ann. Math. Statist.* (1958) **29**, 188-191] and this note provides a further alternative proof which entails little computation.

(R. L. Anderson)

MICKEY, R. (General Analysis Corp., Los Angeles) 3.9 (5.9)

Some bounds on the distribution function of the largest and smallest roots of normal determinantal equations—*In English*

Ann. Math. Statist. (1959) **30**, 242-243 (4 refs.)

While the joint density functions of the roots of certain determinantal equations have been obtained, they are complicated expressions and the marginal distribution functions remain untabulated. This note presents a lower bound for the distribution function of the smallest root and an upper bound for the distribution function of the largest root. These bounding values may be of use in significance tests since observed values which are not " significant " in relation to these bounding values will certainly not be so with respect to the exact distribution.

(R. L. Anderson)

PILLAI, K. C. S. & BENTEGUI, CELIA G. (Statistical Center, Univ. of the Philippines) 3.9 (11.1)

On the distribution of the largest of six roots of a matrix in multivariate analysis—*In English*

Biometrika (1959) **46**, 237-240 (16 refs., 2 tables)

Tables are given of approximate values of the upper five and one per cent. points of what Foster and Rees call a generalised beta distribution $I_x(s;\ m+1,\ n+1)$ for $s = 6$, $m = 0(1)4$ and $n = 5(5)30,\ 40(20)100(30)160,\ 200,\ 300,\ 500,\ 1000$. The method used is to express approximately the exact cumulative distribution. This is carried out by means of a weighted sum of simple beta-variable probability distribution functions. The formula for this is given, but not its derivation, which is described as being by means of the previously published methods of Pillai (Mimeo Series No. 88, Inst. of Statistics, North Carolina) used for the lower values of s. The approximate probability distribution function being evaluated near the true value (estimated by extrapolating from tables for lower s), the tabulated values were obtained by inverse interpolation.

The error, in probability, occasioned by use of these approximate values is discussed. It is stated that for $s = 2,\ 3,\ 4$, the one per cent. points are relatively more accurate; that the accuracy decreases slowly with increasing n (for given m); and that the five per cent. points give an accuracy in probability to within 0.0001. This result is based on earlier results—some in unpublished theses to which references are given in the paper. Arguing from this, and from a comparison at $s = 6$, $m = 0$, $n = 500$ where the error in probability was 0.00004 (at the five per cent. point), it is suggested that the error in probability occasioned by use of the tables is negligible.

Detailed reference to earlier papers is given for the application of the tables.

(D. E. Barton)

PILLAI, K. C. S. & SAMSON, P. Jr. (Statistical Center, University of the Philippines) 3.9 (2.7)
On Hotelling's generalisation of T^2—*In English*
Biometrika (1959) **46**, 160-169 (10 refs., 9 tables)

The authors' introduction states: "Let S_1 and S_2 denote the matrices consisting of the sums of squares and products of deviations from means for two independent samples of sizes n_1 and n_2 respectively, taken from p-variate normal populations. Then T_0^2 may be defined by

$$T_0^2/(n_2-1) = \text{trace } S_2^{-1}S_1$$

so that $T_0^2/(n_2-1)$ equals the sum of the characteristic roots of $S_2^{-1}S_1$."

The authors introduce a statistic $U^{(s)} = T_0^2$, where s equals the number of non-zero characteristic roots of $S_2^{-1}S_1$ (so that $s = p$ if $n_1, n_2 > p$). Expressions are obtained for the first four central moments of the distributions of $U^{(s)}$ for $s = 2, 3, 4, 5$ and 6. Tables of upper 5 and 1 per cent. points of the distributions $U^{(2)}$, $U^{(3)}$, and $U^{(4)}$, calculated by fitting Pearson-type curves, are given for $m = *(0.5)5(5)30, 50$ and $n = *(5)50, 60, 80, 100$ where $m = \frac{1}{2}(n_1-s-2)$, $n = \frac{1}{2}(n_2-s-2)$. The * indicates that the lower limits for m and n vary; for $U^{(2)}$ they are 1.0 and 15 respectively, for $U^{(3)}$ they are 0.0 and 15, and for $U^{(4)}$ they are -0.5 and 10.

(N. L. Johnson)

RAMACHANDRAN, K. V. (Demographic Training and Res. Centre, Bombay) 3.3 (3.1)
On the studentised smallest chi-square—*In English*
J. Amer. Statist. Ass. (1958) **53**, 868-872 (6 refs., 9 tables)

The author is concerned with the distribution of the smallest F ratio in a set of k F-ratios obtained by dividing each of k independent mean squares s_i^2, $i = 1, \ldots, k$, each based on t degrees of freedom by a common (error) mean square based on m degrees of freedom. Whilst the distribution for the largest F-ratio in such a set has numerous applications in analysis of variance testing, the applications of the smallest F-ratio (whose lower percentage points are only of interest) are more restricted, but do arise in certain problems which are mentioned in this paper. Lower 5 per cent. points of the above statistic are tabulated for the following combinations of the essential parameters:

$k = 1(1)8$; $m = 5, 6, 8, 10, 12, 20, 24, \infty$; $t = 1(1)6$
and also for the combinations:

$$t = 1(1)4(2)12(4)20 \begin{cases} m = 5, 6(2)12, 20, 24, \infty \\ k = 1, 2, 3 \end{cases}$$

$$\begin{cases} m = \infty \\ k = 1(1)8. \end{cases}$$

The accuracy is usually three decimals and sometimes more decimals to give at least one significant figure.

(H. O. Hartley)

RIDER, P. R. (Wright Air Dev. Center, Dayton, Ohio) 3.6 (2.5)
Quasi-ranges of samples from an exponential distribution—*In English*
Ann. Math. Statist. (1959) **30**, 252-254 (7 refs.)

This note derives the distribution of the rth quasi-ranges, $W_r = X_{n-r} - X_{r+1}$, where x, $\leqq x_2 \leqq x_3 \leqq \ldots \leqq x_n$ are drawn at random from an exponentially distributed population. It also gives the moment generating function and the cumulants. From these it is shown that the mean of the quasi-range W_r slowly diverges with the sample size while the variance approaches a finite value: for example, in the case of the range, W_0, the variance approaches the value $\pi^2/6 = 1.6449$.

(R. L. Anderson)

ST. PIERRE, J. (University of Montreal) 3.8 (11.1)
Distribution of linear contrasts of order statistics—*In English*
Ann. Math. Statist. (1958) **29**, 1264-1268 (7 refs., 3 tables)

The author considers linear contrasts of n ordered sample values under the assumption that the underlying distributions are normal with unit variances but unknown means $\mu_0, \mu_1, \ldots, \mu_{n-1}$. He presents results for the case of three sample values and a single contrast

$$z = x_{(0)} - cx_{(1)} - (1-c)x_{(2)},$$

where the $x_{(1)}$ are the ordered sample values, $x_{(0)} > x_{(1)} > x_{(2)}$. The density function for z is denoted by $g(z)$. Values of $g(z)$ are tabled for eight choices of c in three cases

(i) $\mu_0 = \mu_1 = \mu_2 = 0$
(ii) $\mu_0 = 1, \mu_1 = \mu_2 = 0$
(iii) $\mu_0 = 2, \mu_1 = 1, \mu_2 = 0$.

The table covers $c = 0, \cdot1, \cdot2, \cdot4, \cdot6, \cdot8, \cdot9, 1\cdot0$ and give the value of $g(z)$ to five decimal places for $z = 0(\cdot2)4$.

The cumulative distribution of z which would be useful for testing hypotheses or for forming confidence intervals is not directly available in this note.

(F. J. Massey)

SANKARAN, M. (Presidency College, Madras) **3.2 (2.2)**

On the non-central chi-squared distribution—*In English Biometrika* (1959) **46**, 235-237 (4 refs., 1 table)

This paper develops the Wilson-Hilferty type of transformation of the non-central chi-squared distribution with f degrees of freedom and non-centrality parameter λ given in 1954 by Aty [*Biometrika*, **41**, 538-540]. Whereas Aty took the cube root, this paper takes the hth power, where h is so chosen as to make the skewness of the resulting variable zero to the first order in $1/f$; that is to say where

$$h = 1/3+2\lambda^2/3(f+2\lambda)^2.$$

Following Aty, the normal approximation and the Fisher-Cornish form of the Edgeworth normalising expansion to the third power of $1/\sqrt{(f+\lambda)}$ are used—the necessary expansions of the first four cumulants of the transformed variable being given.

Approximations to the upper and lower five per cent. points of non-central chi-squared, as obtained by these methods (the second only for the lower point) for $f = 2, 4, 7$ and $\lambda = 1, 4, 16, 25$, are tabulated together with the exact values and Aty's " closer approximation ". The accuracy of the approximation varies according to the values of f and λ and the percentage point considered. For the lower point Aty's value, which gives two or three decimal accuracy, is rather better. For the upper point, whereas Aty's figures differ by two or three in the second decimal place (and in the first for $f = 2$, $\lambda = 1$, being closer for larger f and λ) the present normal approximation is better than both Aty's and the present Edgeworth form with but one exception. This exception is for the case $f = 2$, $\lambda = 1$ when the Edgeworth value of 8.66 is nearer to the true value of 8.64, as compared with the normal approximation of 8.58 and Aty's value of 8.38.

(D. E. Barton)

205

SAW, J. G. (University College, London) **3.8 (4.3)**

Estimation of the normal population parameters—*In English Biometrika* (1959) **46**, 150-159 (3 refs., 3 tables)

It is assumed that, of a complete sample of n taken from a normal population with mean μ and standard deviation σ, only the smallest r observations are available for statistical purposes.

A linear estimator, μ^* for μ is proposed which is symmetrical in the smallest $r-1$ observations. If the estimator is to be unbiased for μ it is necessary to calculate one weight only, the value of which may be shown to be a function of r and n. Values of this weight ($\mathscr{E}$) are given for all (r, n) satisfying $1<r<n\leqq 20$. In addition a series expansion for $\mathscr{E}$ in terms of powers of $1/(n+1)$ is given which will enable the weight to be calculated for larger values of r and n which satisfy $0.20\leqq r/(n+1)\leqq 0.80$. Exact and series results for the variances of μ^* are given over the same ranges. The asymptotic efficiency of μ^* varies between 95 per cent. and 99 per cent. as $r/(n+1)$ varies between 0.20 and 0.80.

A quadratic estimator η^* for σ^2 is proposed which is also symmetrical in the smallest $r-1$ observations. If η^* is to be unbiased and of minimum variance over all such quadratic estimators, it is found that the construction of η^* depends on two weights which are each functions of r and n. Power series expansions are given for these weights and for the variance of η^* for values of r and n satisfying $0.50\leqq r\leqq 0.80$. It is found that the asymptotic efficiency of η^* is 100 per cent. for all values of the ratio $r/(n+1)$.

(J. G. Saw)

206

WEISS, L., (Cornell University)
The limiting joint distribution of the largest and smallest sample spacings—*In English*
Ann. Math. Statist (1959) **30,** 590-593 (1 ref.)

3.8 (2.6)

A limiting joint distribution is presented for the largest and smallest lengths of intervals between ordered values of n independent chance variables, each with the same distribution. The distribution considered is

$f(x) = a_j$ over the interval $[z_{j-1}, z_j)$, $j = 1, 2, \ldots, H$, except that the last interval is $[z_{H-1}, z_H]$.

(R. L. Anderson)

DeGROOT, M. H. (Carnegie Institute of Technology)
Unbiased sequential estimation for binomial populations—*In English*
Ann. Math. Statist. (1959) **30**, 80-101 (18 refs.)

4.7 (4.3)

The author considers first the selection of an appropriate sampling plan for observations on the random variable u, $Pr\{u = 1 \mid p\} = p$, $Pr\{u = 0 \mid p\} = 1-p$, $0<p<1$, to estimate a given function $g(p)$. An estimator f is called efficient at p_0 if the variance of f at p_0 is not greater than that of any other estimator with the same expectation, and associated with a procedure having the same average sample size. If a sampling plan admits an estimator which is efficient at all values of p, then the plan is called efficient.

It is shown that, of all possible sampling plans, only the single sample plans (fixed sample size) and the inverse binomial sample plans (sampling to proceed until a pre-set number of successes or failures has been realised) are efficient in the above sense. In addition, and associated with the former of these plans only, functions of the form $a+bp$ are estimable efficiently; while with the latter, only those of the form $a+b/p$. For these two classes of plans unbiased (but not efficient) estimators are also discussed.

A relation between $g(p)$, p, p_0, and the average sample size, which guarantees that g is efficiently estimable at p_0, is derived. However, the author points out that a general characterisation of such efficiently estimable functions is still an unsolved problem. A discussion of two sampling plans which are combinations of fixed sample and inverse binomial sample plans is given. A study is made of the set of sampling plans by which a given function $g(p)$ may be efficiently estimated at p_0. The author suggests the selection of that plan and estimator which has minimum variance not only at p_0, but " near " it; this turns out to be equivalent to selecting the plan for which the variance of N (sample size) is minimised at relevant values of p. This problem is unfinished, since the existence of such a minimum has not yet been demonstrated. If the set of sampling plans considered has infinitely many members, some restricted results are presented. A new characterisation of completeness is given for bounded plans; this leads to a simple determination of the dimension of the linear space of unbiased estimators of zero for a given plan. The concept of a uniformly minimum variance estimator, useful for a wide class of sampling plans, is introduced and discussed.

(D. N. Shaffer)

DUNN, OLIVE J. (University of California, Los Angeles)
Estimation of the medians for dependent variables—*In English*
Ann. Math. Statist. (1959) **30**, 192-197 (2 refs., 1 table)

4.4 (4.8)

A non-parametric method is used to estimate the unknown medians, ν_1 and ν_2, of two dependent variables, y_1 and y_2. It is only assumed that the two marginal distributions are continuous.

Confidence intervals of the classic type using order statistics are formed separately for ν_1 and for ν_2. If the variables y_1 and y_2 were independent, these two intervals could be used as simultaneous confidence intervals for ν_1 and ν_2 by simply multiplying the two probabilities to obtain the new confidence level. In this paper it is proved that when y_1 and y_2 are dependent, these same intervals may be used as a set with bounded confidence level (as opposed to a set with exact confidence level).

If the sample observations are denoted by

$$(y_{11}, y_{21}), \ldots (y_{1n}, y_{2n}),$$

then for $i = 1$ and 2, the y_{ij} may be reordered from smallest to largest and renamed $z_{i1}, \ldots, z_{in}$. Then $z_{i1} \leqq z_{i2} \leqq \ldots \leqq z_{in}$ for $i = 1, 2$, and it is noted that z_{1j} and z_{2j} need not belong to the same observation. It is proved that for any two positive integers, r and s, such that $2r+s = n$.

$$P\left[z_{1r} < \nu_1 < z_{1,\,r+s+1},\ z_{2r} < \nu_2 < z_{2,\,r+s+1}\right] \geqslant \left[\sum_{j=r}^{r+s} \binom{n}{j} (\tfrac{1}{2})^n\right]^2 = 1-\alpha, \text{ say.}$$

Thus z_{1r} to $z_{1\ r+s+1}$, z_{2r} to $z_{2\ r+s+1}$ are simultaneous confidence intervals for ν_1 and ν_2, and the bounded confidence level is $1-\alpha$.

Comparisons are made on the basis of average length between these intervals and other joint intervals for the means of a bivariate normal distribution.

It is shown by means of a counter-example that the method for obtaining confidence intervals established in this paper cannot be directly extended to three or more variables.

(Olive J. Dunn)

FAVA, L. (University of São Paulo) 4.9 (6.1)

On the variance of the regression-estimate in sampling without replacement from a finite population—*In English*

Bol. Soc. Mat. São Paulo (1959) **10**, 121-128

In sampling surveys, very often we may estimate the mean or the sum of a random variable y, taking advantage of the supplementary information provided by the known mean of another variable x, assumed to be highly correlated with y. The simplest case deals with the estimation of the mean of y when the relation between y and x is assumed to be linear. In this case we use the so-called regression-estimate. In the calculation of the variance of this estimate two procedures are in use:

(i) the sample is assumed large enough so that the proportion of the sampling variation of the regression coefficient to the fluctuation of the estimate can be considered as negligible.

(ii) use is made of the classical regression model, where the population is assumed infinite and the relation between y and x linear, with a random component z statistically independent of x, with mean zero and constant variance.

In this paper, following a suggestion of W. G. Cochran, the author considered the finite population of N sampling units as a random sample of an infinite super-population where the variable y and x were related as given by the classical regression model. Then, in a random sample of size $n<N$, selected without replacement from this population, the x's and the z's are statistically independent, but the x's between themselves and the z's between themselves are not. Under this condition it is proved that the regression-estimate $\tilde{y}$ is an unbiased estimator of the mean of the population values of y. It is also proved that the variance of $\tilde{y}$ (for a fixed sequence of values of x) is affected only in the term independent of x, which must be multiplied by the finite population correction $(1-f)$, where f is the sampling fraction. The sampling variance of z may be estimated from the analysis of variance table with $n-2$ degrees of freedom. The process can be applied also to stratified and multi-phase sampling and to the multiple-regression estimate.

(L. Fava)

FISHER, Sir RONALD (Cambridge, England) 4.0 (4.2)

Mathematical probability in the natural sciences—*In English*

Metrika (1959) **2**, 1-10

In this paper, originally delivered to an International Congress of Pharmaceutical Sciences, Fisher restates his view of the theory and application of probabilities.

He starts with a brief historical review and states the analogy between the two concepts of frequency distributions (as used by Quetelet) and of random variables—although there is no logical identity. Therefore we may make probability statements if our uncertainty is just as though the subject in our mind had been chosen at random from a population with a frequency distribution. Yet this population need not exist but in our imagination, e.g. all throws with a certain die.

It is a statement on a singular and unique subject; for instance, as in the case of Gauss' application of the theory of errors to the measurement of the distance of the sun. Of course there is only one such distance in question but our knowledge enables us to make only a probability statement. Any additional data would change the probability distribution which depends on the premises. Fisher's point of view is equal to that of those writers who ascertain that probability statements always are concerned with syllogisms.

In his mind there are just three requirements for correct statements of probability:

1. the population of possibilities with a frequency distribution.
2. the knowledge that the subject of the statement belongs to the set which may be conceived as this population.
3. the impossibility of recognising a subset which might be expected having different probability distribution. These conditions are necessary and sufficient for all kinds of probabilitity statements. Therefore fiducial probability is not logically distinct from any other kind of probability.

The fiducial argument is described using the example of Student's t and it is stated that the third requirement mentioned above is only fulfilled as the statistics involved (x and s) are exhaustive. This marks one difference between the fiducial method and that of confidence intervals. The latter is not restricted to cases where exhaustive estimation is possible and furthermore it has a quite different meaning since it does not yield probability statements about the parameters.

Fisher also discusses Bayes' theorem and the use of probabilities *a priori*. Bayes did not assume the existence of such probabilities axiomatically, but introduced them as a result of an auxiliary experiment which provides that the parameter can be regarded as a random variable. Fisher shows that another kind of auxiliary experiment has some logical advantages and provides for more general families of distributions. Finally, he discusses briefly the existence of cases where probabilities *a priori* can be assumed to be involved in the problem and states that the commonest case, where they do not exist, can be treated by means of the fiducial argument.

(K. Weichselberger)

GAFFEY, R. (University of California, Berkeley) 4.1 (0.5)

A consistent estimator of a component of a convolution—*In English*

Ann. Math. Statist. (1959) **30**, 198-205 (7 refs.)

Let x_1 represent the measured serum cholesterol determination of a human being randomly selected from some population, while y_i is the individual's true serum cholesterol value. It is assumed that x_i equals the true value of y_i plus an "instrumental error" z_i, i.e. $x_i = y_i + z_i$. It is further assumed that the instrumental errors z_i are normal with mean zero and known variance σ^2. The problem studied is that of estimating the distribution function of the y_i, $G(y)$ say.

The estimator derived is based on a formula due to Pollard which gives $G(y)$ as an infinite series involving all even derivatives of $F(x)$, the distribution of the x_i. The estimator is found by replacing derivatives by finite differences of the empirical distribution function and by truncating the infinite series at the $(n+1)$-st term. This estimator is shown to be consistent; in fact, a formula is derived for the asymptotic mean square error. A considerable amount of algebra is involved.

The author also considers the question whether the instrumental error z_i should be ignored; in which case, $F_n(x)$, the empirical sampling distribution, is used to estimate G. A condition is given when the longer procedure outlined above has smaller bias than the bias of ignoring z_i for the case that $G(y)$ is normal.

It is further noted that the problem of instrumental error has had a long history in astronomy. Several estimates have been given but these appear not to be consistent.

(D. G. Chapman)

213

GART, J. J. (Virginia Polytechnic Institute) 4.5 (4.1)

An extension of the Cramér-Rao inequality—*In English*

Ann. Math. Statist. (1959) **30**, 367-380 (21 refs.)

This paper presents a lower bound for the mean square error of an estimate of a parameter which appears in a frequency function where, instead of the usual assumption of constant parameters, the case where the parameters are random variables is considered. The lower bound presented for the mean square error of an estimate takes into account the variability due to both the observations and the parameters involved and is obtained by extending the Cramér-Rao inequality to this new situation. Necessary and sufficient conditions for equality of the extended inequality are derived.

Most unfavourable distributions of the parameters, i.e. distributions which maximise the lower bound, are discussed, and several examples are given. For the binomial distribution, such a most unfavourable distribution is $G^*(\theta) = 0$ for $\theta < \frac{1}{2}$ and $G^*(\theta) = 1$ for $\theta \geqq \frac{1}{2}$. The Poisson, gamma and normal distributions do not have any most-unfavourable distributions unless some restrictions on the ranges of the parameters are imposed; e.g. for the normal distribution a most-unfavourable distribution cannot be cited unless one can restrict $\sigma^2 \geqq a$, say, and then the most-unfavourable distribution is $G^*(\sigma^2) = 0$ for $\sigma^2 < a$ and $G^*(\sigma^2) = 1$ for $\sigma^2 \geqq a$.

An extension, analogous to that of Bhattacharyya is presented, which provides more stringent inequalities: also the modification of the Wolfowitz extension of the Cramér-Rao inequality for the situation where the sample size is a random variable depending on the sequence of observations. In addition, lower bounds on the variance of linear unbiased estimates of the mean of the parameter are given.

All these extensions for the situation where the parameters are random variables require analogous regularity conditions as are required by their well known counterparts.

(J. W. Wilkinson)

214

KÜHNE, P. (Univ.-Strahleninstitut, Berlin) 4.5 (3.2)

On the estimation of clinically-material differences by means of a " prospective response value "—*In German*

Biom. Zeit. (1959) **1**, 81-87 (3 refs., 1 table, 2 figs.)

In clinical experiments—for example, testing the effect of a medical treatment—increasing the sample number leads to the detection of progressively smaller mean differences δ which are (below a certain unspecified value) therapeutically irrelevant since the expected ratio $p/(1-p)$ of " responding " ($\delta > 0$) to " non-responding " ($\delta \leqq 0$) individuals tends to unity (in case of the normal distribution considered here). Under this aspect the usual significance tests are no longer appropriate. The question for therapeutically relevant δ's is reframed in terms of a " prospective response value " Δ, which is the standardised mean difference (δ/σ) corresponding to a given lower value of p. The author avoids the exact solution of the problem by equating the lower $100\alpha_\delta$ per cent.-fiducial level for δ to the upper $100\alpha_\sigma$ per cent.-fiducial level for σ, obtaining a kind of rough " lower simultaneous fiducial level " for Δ corresponding to a given p, i.e. a lower critical value D_{min} for the empirical standarised difference $D = d/s$. A table for D_{min} for $\alpha_\sigma = 0.95$ and 0.99, and $\alpha_\sigma = 0.95$, $p = 2/3$, 19/20, 99/100 is given for degrees of freedom $= 1(1)30(10)60(20)120, \infty$.

Lacking a rigorous mathematical treatment, no exact fiducial levels for D_{min}, given p (or conversely) are obtained. This disadvantage is supposed to be compensated by the fact that only readily accessible tables are required.

(R. Wette)

215

KULLDORF, G. (University of Lund, Sweden) 4.1 (4.2)

On the conditions for consistency and asymptotic efficiency of maximum likelihood estimates—*In English*

Skand. Aktuartidskr. (1957) **40**, 129-144 (2 refs.)

The discussion of the asymptotic properties of maximum likelihood estimates has followed two separate lines in the literature. Some authors, among them Cramér [*Mathematical Methods of Statistics*, pp. 500-504], have studied the root of the likelihood equation, while others have directly considered the parameter value providing the absolute maximum of the likelihood function. This paper follows the first of these lines.

Three theorems are presented, each one giving sufficient conditions for consistency and asymptotic efficiency in the strict sense, respectively. Theorem 1 is almost identical with Cramér's original theorem. Theorem 2 is a direct generalisation of Theorem 1, in that one of the conditions is replaced by a weaker one. Theorem 3 contains an alternative set of conditions: in contrast to Theorems 1 and 2 it does not assume that the third partial derivative of the density function with respect to the parameter exists, and the consistency property does not even require the existence of the second partial derivative.

It is also found that the conditions for consistency in either theorem imply that, with probability tending to unity as the sample size tends to infinity, the root of the likelihood equation exists and provides the absolute maximum of the likelihood function.

(G. Kulldorf)

216

LLOYD, D. E. (Ordnance Board, Ministry of Supply, England) 4·3 (3.8)

Note on a problem of estimation—*In English*

Biometrika (1959) **46**, 231-235 (1 fig.)

A random variable r is assumed to have a frequency distribution function $p(r)$ which may contain unknown parameters. Further, the probability that r is "observable" is given by a frequency function $F(r)$ which may also contain unknown parameters.

The author writes down the log likelihood function, differentiates with respect to a single unknown parameter and equates to zero:

(*a*) when only the number of observable values (m) together with the initial sample size (n) are given,

(*b*) when the numerical values of the m observations from a sample of size n are given. Here the author considers a likelihood function conditional on there being m recorded results.

As a specific case, it is assumed that r is distributed as an unknown multiple of a chi-squared variate with known degrees of freedom; and that $F(r)$ is a known negative exponential distribution. It is required to estimate the scale parameter. The maximum likelihood solution is given in an implicit form. The particular case of two degrees of freedom is considered in a little more detail.

(J. G. Saw)

SCHMETTERER, L. (University of Hamburg) 4.6 (5.6)

On nonparametric methods in mathematical statistics—*In German*

Jber. Dtsch. Mathver. (1959) **61**, 104-126 (43 refs.)

This paper gives a concentrated survey on the abstract theory of non-parametric methods. The results are cited without proof.

(A) *Estimation of Parameters*:

Let K be a specified class of probability measures defined on a measurable space (R, b) and $m(P)$ a functional defined for every $P \in K$. A real measurable function $h(x)$ $(x \in R)$ is called an unbiased estimator of $m(P)$, if $E(h(X); P) = m(P)$. The subsequent results are valid for loss functions of the type $G(h(X) - m(P))$, where G is non-negative and strictly convex. Under this assumption, there exists at most one unbiased estimator with minimum risk (Lehmann-Scheffé). If there exists a statistic $T(x)$ sufficient for K, then $E(h(X) \mid T)$ is an unbiased estimator of $m(P)$ with a risk function not greater than the risk function of $h(x)$ for all $P \in K$. (Equality holds if and only if $h(x)$ can be expressed as a function of $T(x)$). Therefore, if H is the class of all unbiased estimates of $m(P)$, then the class of unbiased estimators $\{E(h(X) \mid T),\ h(x) \in H\}$ is complete. If the statistic $T(x)$ is complete the unbiased estimator with minimum risk is unique (Lehmann-Scheffé). Therefore, if $T(x)$ is a complete sufficient statistic and $h(x)$ an unbiased estimator, then $E(h(X) \mid T)$ is an unbiased estimator with minimum risk.

In the following, the measurable space is assumed to be a cartesian product and K a class of product measures. Then, the order statistic $t(x)$ is sufficient. As the functions $E(h(x_1, \ldots x_n) \mid t)$ are symmetric, it follows from above, that the class of symmetric unbiased estimators is complete. $t(x)$ is not only sufficient, but also complete, if the underlying class of measures is the class of all absolutely continuous or the class of all discrete distributions. Therefore, in these cases the unbiased estimator with minimum risk is unique and necessarily symmetric.

(*B*) *Tests of Hypotheses*:

Let k be the subset of K, called the null hypothesis. A group $\mathcal{G}$ of transformations of R is called admissible, if the induced group of transformations of K is such, that the subsets k and $K - k$ are closed under $\mathcal{G}$. A test function is called invariant if it is invariant under any admissible group $\mathcal{G}$. If a sequence of asymptotically invariant probability measures on $\mathcal{G}$ is given, then an invariant minimax test exists. Restricting to $R = R_n$ (euclidean space), a theorem of Lehmann is given concerning invariant (under continuous and strictly monotone transformation) tests which are uniformly most powerful against a specific class of alternatives. This theorem is of special importance in connection with the two-sample problem: the contributions of Wilcoxon, Terry, van der Waerden, and Smirnoff are cited.

The restriction to continuous and strictly monotone transformations which let the element o invariant leads to median tests, which contain the sign test as a special case. The further restriction to continuous and strictly monotone transformation, symmetric around o, leads to the symmetric tests. Finally, the concept of unbiased tests is introduced and a theorem by Sverdrup is cited.

(J. Pfanzagl)

SCHUTZENBERGER, M. P. (University of Poitiers) 4.2 (–.–)
Concerning the Fréchet-Cramér inequality—*In French*
Publ. Inst. Statist. Paris (1958) 7, 3-6

In this note the author shows how the classical inequality is transformed into the case of Bayes estimation. That is to say the case where the parameter to be estimated has a probability density *a priori*.

(D. Dugué)

SCHWARTZ, LORRAINE & WEARDEN, S. (Kansas State Univ.) 4.6 (5.6)
A distribution-free asymptotic method of estimating, testing, and setting confidence limits for heritability—*In English*
Biometrics (1959) **15**, 227-235, (7 refs., 1 table)

Detection of the heritability parameter is often the chief purpose of genetic experimentation. Several different procedures are available to estimate heritability when the observable random variables can be assumed to be normally distributed. There are, however, characteristics of interest to the geneticist whose distributions are known to be non-normal.

In this paper a rank-order statistic is defined whose expected value is taken as a measure of heritability. This statistic requires that the parental and first generation can be ranked on the basis of the characteristic under study. The statistic is then taken as proportional to the ratio of the difference in mean ranks of the upper and lower groups of the parental generation, to the difference in mean ranks of the upper and lower groups of the first generation. It is shown that this statistic is related to the Mann-Whitney U statistic, and this relation is utilised to estimate the heritability of the characteristic, to test the estimate for significance, and to place confidence intervals on the parameter. An example is given in which the characteristic under study is the rank in the peck order of a flock of hens.

(C. P. Quesenberry)

BARRACLOUGH, ELIZABETH D. & PAGE, E. S. (Durham Univ. Computing Laboratory) 5.7 (11.1)
Tables for Wald tests for the mean of a normal distribution—*In English*
Biometrika (1959) **46**, 169-177 (9 refs., 3 tables, 1 fig., 4 charts)

Tables and charts are provided for obtaining the constants of Wald's sequential test for an unknown mean when the standard deviation is known. Tables are also given for the average sampling numbers.

If an observation x is a normally distributed random variable with known standard deviation and unknown mean θ, the test for the hypothesis that $\theta = \theta_0$ against the alternative that $\theta = \theta_1$ may be specified thus:—continue sampling as long as a certain linear function of the observations lies between $-Z$ and $h-Z$; if the function first becomes less than $-Z$ stop sampling and accept that $\theta = \theta_0$, similarly accept that $\theta = \theta_1$ if the function first becomes greater than $h-Z$.

Denoting the probabilities of accepting that $\theta = \theta_0$ by P_- when $\theta = \theta_0$ is true and by P_+ when $\theta = \theta_1$ is true, the values of h and Z are given as entries in a two way table with arguments P_- (4 values, ·95 to ·999) and P_+ (8 values ·05 to ·70) for each of four values of the standardised difference between θ_0 and θ_1. Charts are provided as alternatives to this table. The average sampling numbers N_- (when $\theta = \theta_0$) and N_+ (when $\theta = \theta_1$) are given as entries in a similar table.

The second section of the paper describes uses of the tables, with examples, while section three gives the iterative method used in computing the tables.

In the final section, Wald's approximation for the operating characteristic of the test is examined for various values of h, Z, and the standardised difference of θ_0 and θ_1.

(N. W. Please)

BARTHOLOMEW, D. J. (University College of North Staffordshire) 5.4 (3.1)
A test of homogeneity for ordered alternatives—*In English*
Biometrika (1959) **46**, 36-48 (4 refs., 8 tables)

Given k normal variates with mean values $m_1, m_2, \ldots, m_k$, a test is suggested for the hypothesis H_0 that these means are equal, against the alternative hypothesis H_2 that $m_1 \geqq m_2 \geqq \ldots \geqq m_k$ where at least one of the inequalities is strict.

Two test criteria are given, one when the variances of the variates are known and the other when they are estimated from the data as in a one-way analysis of variance by groups. These criteria are functions of estimates of the mean values which are obtained thus:

> Arrange the observations in the order predicted by H_2. If any consecutive pair are not in the expected order replace them by their average, reducing the number in the series by one. Repeat this process, treating an average of two as a single observation but with weight two, etc., until the resulting values are in the order predicted under H_2.

The distributions, under H_0, of the two criteria are obtained formally as functions of the χ^2 and F distributions respectively and also of $P\{l, k\}$, the probability that the reduced series consists of l means when there were originally k observations. The determination of $P\{l, k\}$ is essential. Complete results for unequal variances are given only for $k = 3$ and 4, with a partial result for $k = 5$. A recurrence relation is conjectured when the variances are equal.

Five and one per cent. points of the first criterion are tabulated for $k = 3$ and 4 for a number of combinations of values for the variances of the variates. Three examples of the use of the tests are worked through.

An editorial note, added in proof, points out that R. E. Miles has now proved the recurrence relation conjectured in this paper and has established a more useful recurrence for $P\{l, k\}$ which can be used for testing the significance of the two criteria for $k > 5$ when the variances are equal.

(N. W. Please)

BHATE, D. H. (Univ. Coll., London & Delhi School of Economics) 5.7 (5.2)

Approximations to the distribution of sample size for sequential tests. I. Tests of simple hypotheses—*In English*

Biometrika (1959) **46**, 130-139 (10 refs., 5 tables, 1 fig.)

Some simple approximations to the distributions of sample size in standard sequential tests are developed. A lower bound for the probability of termination at, or before, the mth stage is obtained from the probability that the sample point for a (fixed size) sample of size m falls outside the continuation region. An upper bound is obtained by adding to this a quantity which exceeds the probability of reaching a decision before the mth stage but subsequent observed values pulling the sample point back into the continuation region.

Numerical results are compared with exact values and with the results of sampling experiments. Some further approximations are obtained on the assumption that the sample size distribution is the same when either of the two hypotheses on which the sequential test is based is true. A brief investigation of the adequacy of this assumption is given.

(N. L. Johnson)

CHERNOFF, H. & SAVAGE, I. R. (Stanford Univ. and Univ. of Minnesota) 5.6 (3.8)

Asymptotic normality and efficiency of certain nonparametric test statistics—*In English*

Ann. Math. Statist. (1958) **29**, 972-994 (13 refs.)

Sufficient conditions for the asymptotic normality of many nonparametric test statistics comparing two populations have been established in earlier papers. This paper extends these results to cover more situations.

The two populations are characterised by absolutely continuous cumulative distribution functions $F(x)$ and $G(x)$. Random samples of size m and n, respectively, are obtained from these two populations; these $N = m+n$ observations are ordered from low to high. Let E_{Ni} be a number assigned to the i-th smallest observation ($N1$ being the smallest and NN the largest). The authors consider a statistic (T_N) which is the average value of the E_{Ni} for those observations in the F-group (sample of size m). Tests differ in the method of determining the E_{Ni}. For example, for the Wilcoxon test $E_{Ni} = i/N$, where E_{Ni} is the normalised score (mean deviation) for ranked data used in the c_1-statistic advanced by Terry.

If $\lambda_N = m/N$ is bounded (greater than 0 and less than 1) as N increases and with certain other minor restrictions, it is shown that T_N is asymptotically normally distributed. In particular, these results hold when $\mathscr{E}_{Ni}$ is the expected value of the i-th order statistic of a sample of N from the population specified in the test. This latter statement establishes that the c_1-statistic is asymptotically normal. The authors also show that the test for translation using c_1 is at least as efficient as the t-test; the efficiency is unity only when $F(x)$ and $G(x)$ are normal.

This test for translation assumes that F and G have the same form but that $F(x) = \psi(x-\theta)$ and $G(x) = \psi(x-\phi)$. Under the null hypothesis, $\theta-\phi = \Delta = 0$; under certain alternatives, $\Delta = cN^{-\frac{1}{2}}$.

(Olive J. Dunn)

DARWIN, J. H. (Dept. Sci. Industr. Res., Wellington, New Zealand) 5.7 (2.4)
Note on a three-decision test for comparing two binomial populations—*In English*
Biometrika (1959) **46**, 106-113 (3 refs., 1 table, 1 fig.)

The test is constructed in the following way:—suppose an event E can occur in each of two random processes. We take a pair of trials, one from each process, and observe a variable γ which is equal to $+1$ if E occurs in the first, but not the second process, 0 if the two processes give the same result (E or not$-E$) and -1 if E occurs in the second, but not the first process. $\gamma(m)$is defined as the sum of m independent values of γ, with $\gamma(0) = \frac{1}{2}(n+1)$ for some appropriate n. If $\gamma(m)$ crosses the line $\gamma(m) = 0$ before it crosses $\gamma(m) = n+1$, we accept the first process; in the converse case we accept the second process. If neither of these events occur no decision is made.

Tables are given of the values of n and of the sample size M such that the probability of correct acceptance has some specified value.

The author compares his test with one devised by Armitage [*Biometrika* (1957) **44**, 9-26] and shows that the latter test may need more observations, but is less sensitive to error in distinguishing between the choice of binomial proportions used in constructing the tests.

(N. L. Johnson)

DEMPSTER, A. P. (Bell Telephone Laboratories) 5.8 (5.3)
A high dimensional two sample significance test—*In English*
Ann. Math. Statist. (1958) **29**, 995-1010 (9 refs., 1 table)

The author considers testing the difference between the means of two k-variate populations which have the same covariance matrix. When the sample sizes are small ($n_1+n_2-2<k$), the classical method (Hotellings' T^2 statistic) fails, since there are not sufficient observations to define a k-dimensional metric on the basis of the variation of the samples about their means. The equivalent best linear discriminator of Fisher similarly fails. The author assumes there is available a k-dimensional metric which has been determined " apart from the data ".

Using this *a priori* metric, the data are re-expressed as a set of orthogonal vectors. This permits the definition of an F-type statistic. Since the distribution of a non-negative quadratic form is approximately chi-square with r degrees of freedom, this statistic has, approximately, the F distribution (when r is known). Estimates for r are obtained, based on the approximate distributions for the quadratic forms. The author suggests a test procedure based on using an estimate for r in place of the unknown parameter in the approximate distribution of this statistic. The practical utility of this procedure is supported by some numerical computations on a conditional distribution of the significance level of the proposed observed test statistic, and by showing the test to be asymptotically conservative when r is large.

The generality of the original orthogonalisation of the data is discounted. The sensitivity of the test is discussed in terms of the distance between their means, as measured by the natural metric based on the covariance matrix. Generally the *a priori* metric should coincide (up to a scale factor) with the natural metric. The author gives some attention to the effect of a lack of coincidence. The last section describes geometrically the case of r large.

(T. A. Jeeves)

DEMPSTER, A. P. (Harvard University)
Generalised D_n^+ statistics—*In English*
Ann. Math. Statist. (1959) **30**, 593-597 (4 refs.)

5.6 (–.–)

The cumulative distribution function of a sample of n from a uniform distribution on (0, 1) is denoted by $y = F_n(x)$, where the population cumulative distribution function is $y = x$. The author is concerned with a set of probabilistic questions about $F_n(x)$ crossing barriers represented by lines joining the (x, y) points $(0, \delta)$ and $(1-\epsilon, 1)$; these barriers are referred to as (δ, ϵ) barriers. If $\delta = \epsilon = d$, D_n^+ is that d for the farthest barrier reached in the sample. A more general class of barriers are those moving away from $y = x$ but restricted only by the requirement that no two members of the class intersect within the unit square. In addition, some attention is paid to the class of barriers $(0, d)$.

(R. L. Anderson)

GNANADESIKAN, R. (University of North Carolina)
Equality of more than two variances and of more than two dispersion matrices against certain alternatives—*In English*
Ann. Math. Statist. (1959) **30**, 177-184 (8 refs.)

5.4 (5.8)

In this paper, the heuristic union-intersection principle of S. N. Roy is used for the construction of two tests and the associated simultaneous confidence bounds on certain parametric functions. However, since the appropriate tables are at present not available, immediate practical application of the tests and associated confidence bounds is not possible.

The first test is of the composite hypothesis for the equality of the variances for $(k+1)$ univariate normal populations $(k \leqq 1)$ or, equivalently, the hypothesis that the ratios of k of the variances to the remaining "standard" (but still unknown) variance are equal to unity—against the alternative hypothesis that at least one of these variance ratios is different from one. The test proposed (referred to as the Simultaneous Variance Ratios Test) has an acceptance region the intersection of the k well-known intervals which form the acceptance regions for the k separate hypotheses that each of the variance ratios is equal to unity. In the absence of knowledge of an optimum choice for the boundaries of these intervals, the author suggests that they be chosen so that each of the "individual" acceptance regions will have the same size and will provide locally unbiased tests of the individual hypotheses. With this choice of the boundaries, the test is completely unbiased and the power of the test increases monotonically as each of the variance ratios diverges from unity. When the samples are of the same size, the proposed test is not equivalent to Hartley's $F_{\max}$ test for the case of equal sample sizes since the alternatives in the two cases are different.

Simultaneous confidence bounds on the k variance ratios with a joint confidence coefficient $(1-\alpha)$ are presented. With the above indicated choices for the boundaries of the individual regions the given confidence bounds have the property of monotonically decreasing length (in terms of the probability of covering wrong values).

The second test proposed is the multivariate extension of the above to $(k+1)$ non-singular p-variate normal populations. In this case, no investigation for desirable power properties is given. Simultaneous confidence bounds are also presented for the characteristic roots of the products of the k dispersion matrices with the inverse of the $(k+1)$-*st* "standard" dispersion matrix with a joint confidence coefficient greater than or equal to $(1-\alpha)$.

The author mentions that Hartley's $F_{\max}$ test is more involved that his proposed test but that it has a more detailed structure of alternatives. He also indicates that he is currently investigating a generalisation of Hartley's test to the case of unequal sample sizes (as well as a multivariate extension) a discussion of which is planned for a future paper.

(J. W. Wilkinson)

GOOD, I. J. (Government Communications H.Q., Cheltenham) 5.0 (5.8)

Significance tests in parallel and in series—*In English*

J. Amer. Statist. Ass. (1958) **53**, 799-813 (19 refs., 2 tables)

Two or more significance tests arising from different statistics based on the same data and testing the same null hypothesis are said to be " in parallel ". Tests based on independent sets of data are said to be " in series ". Contradictory advice regarding parallel tests are found in the literature. Should one insist on using only a single test, specified in advance, or should one permit several tests, some of which may be suggested by the data ? The latter is said to be " reasonable, dangerous and often done ".

Three situations which suggest the use of parallel tests are:

A. Several non-null hypotheses, some perhaps suggested by the data;

B. Several inefficient tests against a fixed alternative; and

C. Several tests against a fixed alternative, some being efficient and others robust.

The author suggests that in cases A and B " if the tail-area probabilities of several statistics on the same evidence . . . are $P_1, P_2, \ldots, P_n$, then, *if the statistician can think of nothing better to do*, or if to do so would involve too much work, it is reasonable for him to summarise them approximately by using their harmonic mean, $\lambda = n/\Sigma P_i^{-1}$, or a weighted harmonic mean, $\lambda(w) = \Sigma w_i/\Sigma w_i P_i^{-1}$. . . ." The weights w_i are to be supplied subjectively; they are " proportional to the probability that the i-th test is the one we should be using if we were wise enough ".

The use of λ or $\lambda(w)$ has the desirable property that a single small tail-area strongly influences the result. Further justification is given by a Bayesian argument. The " Bayes' factor F_i against the null hypothesis H_0 given the statistic ϕ_i " is $F_i = Pr\{\phi_i(E) \mid \bar{H}_0\}/Pr\{\phi_i(E) \mid H_0\}$ where E is the evidence and $\bar{H}_0$ is the alternative hypothesis. If the Bayes' factor given $\phi_i(E)$ identically equals the Bayes' factor given E, than ϕ_i is called " efficacious for testing H_0 ". The author asserts that empirically F_i is approximately inversely proportional to P_i. Using this approximation, a rule for combining Bayes' factors F_i leads to the weighted harmonic mean for combining tail-areas P_i.

For combining tests in series, Fisher's method is available. A generalisation employs the statistic $P_1^{\alpha_1} P_2^{\alpha_2} \ldots$ whose distribution was previously given by the author. A rule of thumb is now given for choosing the weights α_i.

Four appendices are titled: I. Efficacious statistics; II. Testing the mean of a multivariate normal distribution of known covariance matrix; III. An example concerning the harmonic-mean rule of thumb; IV. The relationship between Bayes' factors and tail-area probabilities. In III, the theory is illustrated by giving individual and combined tail-area probabilities for a sample of 30 from a bivariate normal population.

(R. J. Buehler)

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KIEFER, J. (Cornell University) 5.4 (11.1)

k-sample analogues of the Kolmogoroff and Cramér-v. Mises tests—*In English*

Ann. Math. Statist. (1959) **30**, 420-447 (26 refs., 5 tables)

Tables are given for two limiting distributions. These tables can be used for large-sample tests of either " the homogeneity hypothesis H_1 that k random samples of real random variables have the same continuous probability law, or the goodness-of-fit hypothesis H_2 that all of them have some specified continuous probability law ". The consistency and power of the suggested tests is discussed.

The tests are defined as follows: the independent random variables X_{ji} are samples of size n_j from the k unknown continuous distribution functions F_j. The hypotheses are $H_1: F_1 = F_2 = \ldots = F_k$, and $H_2: F_1 = F_2 = \ldots = F_k = G$, where G is a specified continuous distribution function. The number of X_{ji} (for i between 1 and n_j) which are less than or equal to x is denoted by $n_j \cdot S^{(j)}(x)$, and the sum of these k numbers is denoted by $(\sum_j n_j) \cdot \bar{S}(x)$. The statistics considered for testing H_1 are $T_N = \sup_x V(x)$ and

$$W_N = \int_{-\infty}^{+\infty} V(x) d\bar{S}(x), \quad \text{where} \quad V(x) = \sum_j n_j [S^{(j)}(x) - \bar{S}(x)]^2.$$

The two similar statistics T_N' and W_N', in which, $G(x)$ has been substituted for $\bar{S}(x)$, are used for testing H_2. Each test requires the hypothesis to be rejected if the observed value of the statistic is too large. The author indicates that the power of tests based on criteria like T is better than tests based on criteria like W.

The author demonstrates the following distribution limits, as all n_j tend to infinity:

(1) for testing H_1,

$$P\{T_N \leqq x^2\} \to \Phi_{k-1}(x) \text{ and } P\{W_N \leqq x\} \to B_{k-1}(x);$$

(2) for testing H_2,

$$P\{T_N' \leqq x_2\} \to \Phi_k(x) \text{ and } P\{W_N' \leqq x\} \to B_k(x).$$

Infinite series for $\Phi_h(x)$ and $B_h(x)$ are given. For $h = 1, 2, 3, 4, 5$, six-decimal tables were prepared on a digital computer. These tables use an abscissa increment of 0.01 to cover the ordinate range from 0.001 to 0.999. To facilitate application of the suggested tests, five-decimal tables of the inverse functions are given for the most often used significance levels. There is a four-decimal table of the convolution of Φ_1 with itself.

This paper contains interesting remarks by the author on related matters—other statistics are suggested; alternative methods of proof are described; various possible series are developed; and other criteria, using previous results, are indicated.

(T. A. Jeeves)

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LAWLEY, D. N. (Mathematical Institute, University of Edinburgh) 5.8 (6.4)
Tests of significance in canonical analysis—*In English*
Biometrika (1959) **46**, 59-66 (4 refs.)

The standard reduced situation of canonical analysis is considered, where x_1 and x_2 are $p-$ and $q-$ variate vectors the components of which are jointly multinormally distributed with zero means and unit variances and where the only non-zero correlations are between corresponding components in the first k elements of x_1 and x_2, the k canonical correlation coefficients being taken in descending order of magnitude; $k<p\leqq q$. A sample of n independent observations is taken from this $(p+q)$-variate population and the canonical correlations estimated. It is assumed that the probability that the sample values are not in the population order is negligible. This situation is considered as a null hypothesis situation as against the alternative hypothesis that there are more than k non-zero canonical correlations. The problems treated are the distribution of the sample canonical correlations and of Bartlett's likelihood ratio test function as between the hypotheses described.

The squares of the sample canonical correlations (i.e. including those with zero population values) are the roots of a certain matrix which is expanded into a series of matrices whose elements have expectations in ascending powers of $1/n$. This enables the author to use a result given in an earlier paper expressing a typical one of the first k sample canonical correlations in terms of these elements and thence to obtain the leading terms in its bias and in the second, third and fourth cumulants. The z-transform is considered, especially in the case of the multiple correlation coefficient ($p = k = 1$) and found to be not quite so effective as in the classical case.

The expectation of Bartlett's criterion is obtained to order $1/n$ thus giving a correction in the form of a multiplying factor depending on the population canonical correlations (and taking Bartlett's value when these are all unity); it is suggested that the values estimated from the sample be used.

When x_1 consists of a fixed set of numbers a regression situation is obtained and the same methods are used to get a first-order correction factor for the likelihood ratio test criterion of the hypothesis that there are exactly k non-zero canonical regression coefficients. This is a function of the population values and if their estimates are used the correction factor turns out to be the same as that used in the correlation analysis.

(D. E. Barton)

LIPTÁK, T. (Math. Inst., Hungarian Academy of Sciences, Budapest) 5.0 (5.1)
On the combination of independent tests—*In English*
Magy. Tud. Akad. Mat. Kut. Intézet. Közleményei (1959) **3**, 171-197 (17 refs., 2 figs.)

Fisher's " omnibus test " for combining the results of several independent tests regarding a common hypothesis is well known. This paper deals with the same problem but it gives a more general answer to the question.

The author starts off, as does Fisher, from the " moving levels " of the individual tests. (The moving level is the minimal level for which the actual result is still significant.) The class of available combinations is narrowed down by the following rational postulates:

1. the combination should be monotonic, i.e. a set of levels should be more significant than another if the levels of the former are more significant than the corresponding levels of the other set.
2. the combination should be compatible, i.e. in case of joining different sets of levels the value of the combined level should be uniquely determined.
3. the value of the " combined " level of a set consisting of a single level, should agree with the value of this level.

It is proved that the class of combinations satisfying the mentioned postulates are the combinations generated by the weighted means of the levels, averaging by any continuous and strictly increasing function.

In particular it is shown that the (ordinary) geometrical mean corresponds to Fisher's " omnibus test " while the weighted one to its generalisation introduced by I. J. Good [*J. R. Statist. Soc.* B (1955) **17**, 264-265].

Fisher's " omnibus test " is the likelihood ratio test of the combination problem for a wide class of tests which are in some sense " good " tests.

The author suggests the combination generated by the inverse ϕ^{-1} of the normal distribution function

$$\phi(t) = 1/(2\pi)\int_{-8}^{t} \exp(-u^2/2)du.$$

In this case the combined level $\tilde{\mathscr{L}}$ is computed from formula

$$\tilde{\mathscr{L}} - \phi\{\lambda_1\phi^{-1}(L_1)+\lambda_2\phi^{-1}(L_2)+\ldots + \lambda_r\phi^{-1}(L_r)/\sqrt{\lambda_1^2+\lambda_2^2+\ldots+\lambda_r^2}\}$$

where $L_1, L_2, \ldots, L_r$ denote the moving levels of the tests to be combined. The weights $\lambda_1, \lambda_2, \ldots, \lambda_r$ should be chosen as to express the efficiencies—these are proportional to the " expected " differences between the null-hypothesis and the real situation and inversely proportional to the standard deviations of the statistics used in the experiments. In some cases it is suitable to choose $\lambda_i = \sqrt{n_i}$ where n_i is the number of observations in the ith experiment.

The suggested combination is optimal for a large class of tests for one-sided hypotheses. In addition, its application needs less numerical work and simpler tables than the " omnibus test ".

(K. Sarkadi)

MACHEK, J. (Charles University, Prague) **5.2** (5.7)

On a two-sample procedure for testing Student's hypothesis using mean range—*In English*

Aplik. Mat. (1959) **4**, 211-223 (7 refs. 2 tables)

The author modifies the two-sample method by Stein for testing a hypothesis about the mean of a normal distribution for the case when population variance is estimated by means of sample range or mean range of several samples. In this paper the test of hypothesis that the mean m equals to a given value m_0 against one-sided (say greater) alternatives is studied. For a given value m_1 greater than m_0 a two-stage testing procedure is constructed whose operation characteristics passes through the points (m_0, a) and $(m_1, 1-b)$; a and b are prescribed positive numbers. This is a situation which arises often in quality control, where a and b represent consumer's and producer's risk, respectively.

At the first stage of the procedure, a random sample consisting of k mutually independent subsamples of size n is drawn from the population, and the mean range R of the subsamples is calculated. The subdivision into smaller samples serves to increase the efficiency of the estimate of population variance and to facilitate the identification of extreme values.

The size of the sample drawn at the second stage depends on the observed value of R, on the difference between m_1 and m_0, and on the prescribed risks a and b. The test-statistic is a suitable linear combination of the mean of the first sample and the mean of the second sample divided by R. The distribution of the statistic under null-hypothesis is that of a standardised normal variable divided by an independent variable distributed as the mean range of the sample from normal population with unit variance. To calculate critical points for the test-statistic, the author uses an approximation suggested by Patnaik of the distribution of the variable in the denominator. He supposes that this variable is a constant multiple of the square root of a variable having chi-square distribution but without necessarily an integer number of degrees of freedom. The approximate 5 and 10 per cent. critical points are tabulated for different n and k.

Because the number of observations needed by the procedure is a random variable, the average sample size under the null hypothesis is investigated and approximate expressions of it are given. Two tables are presented. The first illustrates the dependence of the average size on the subdivision of the first sample when the size of the sample is kept fixed. The second table shows the dependence of the expected number of observations on the size of the first sample. The tables are computed by means of approximate expressions mentioned in the paper.

(P. Mandl)

MITRA, S. K. (North Carolina: now at Indian Statist. Inst., Calcutta) **5.2** (5.1)

On the limiting power function of the frequency Chi-square test—*In English*

Ann. Math. Statist. (1958) **29**, 1221-1233 (18 refs.)

While several authors have investigated the power function of the frequency χ^2-test, in particular the simple goodness-of-fit test, there has been no unified treatment of the case where the parameters required for the class frequencies have to be estimated. In 1948, E. J. G. Pitman proposed the concept of asymptotic power. This present paper derives the Pitman limiting power of the χ^2-test when the unknown parameters involved in the specification of the class frequencies are estimated from the sample by a method which is asymptotically efficient. The paper concludes with some discussions of the Pitman-limiting-power for the frequency χ^2-test in connection with (*a*) the planning of experiments for comparing two distribution functions and (*b*) the planning of experiments to detect shifts in response.

After a section stating the concept of limiting power (due to Pitman, 1948), the author states a theorem, whose proof requires five lemmas, that a system of equations has (under certain conditions) exactly one system of solutions and that the limiting value of χ^2 which results is distributed as a non-central χ^2 variate. He then proceeds to consider the limiting power of the frequency χ^2-test in connection with a problem posed by Neyman in 1949 [*Proc. Berkeley Symp. Math. Statist. and Probability*].

(W. R. Buckland)

PFANZAGL, J. (Univ. of Vienna) **5.2** (5.4)

A combined test and classification problem—*In German*
Metrika (1959) **2**, 11-45 (20 refs.)

Most of the statistical text-books contain only the F-test as a test for the comparison of k samples from k different distributions, i.e. a test whether there are differences between means. Tests are not usually given to show where the means differ from each other. As the latter question is more important for many problems several methods for it have been proposed during the last years. However, besides the more comprehensive decision theory of Wald, there is no appropriate generalisation of the theory of Neyman-Pearson. This paper supplies one for the special case when only one distribution is different, and all alternatives have the same probability.

The null hypothesis H_0 and k alternative hypotheses $H_1, \ldots, H_k$ are given. First H_0 is to be tested with the specified significance level α against the composite hypothesis $H_1, \ldots, H_k$ and, if H_0 is rejected, one of the alternatives is to be determined. If H_0 is not valid, it is assumed that each of the k alternatives has the same probability. The decision process is defined as optimal if the sum of the k conditional probabilities of a correct decision given H_i is a maximum.

It is demonstrated that the decision process is optimal if Max $[f_1(x), \ldots f_k(x)]/f_0(x)$ is used as the decision statistic ($f_i(x)$ is the density under H_i). Thus, the optimal decision process substantially consists of a modified likelihood-ratio-test and, if H_0 is rejected, we decide for that H_i of the k alternatives with the greatest likelihood. The theory is applied to the following problems:—

(1) Slippage problem:
H_0: all k of a sequence of random variables $X_1, \ldots, X_k$ have a given distribution. H_i: X_i, and only that, has a different distribution.
For the normal or binomial distribution the decision statistic is the greatest observation, but not for the Cauchy distribution.

(2) Shift problem:
H_0: the same as (1). H_i: only $X_i, X_{i+1}, \ldots, X_k$ have a different distribution.

(3) k sample problem:
H_0: all k samples are from identical distributions.
H_i: only the ith sample is from a different distribution.

In case of a normal distribution for testing the equality of means and variances the author obtains the same results as Paulson [*Ann. Math. Statist.* (1952) **23**, 610-616] and Truax [*Ann. Math. Statist.* (1953) **24**, 669-674]. These results are generalised for the case of samples of unequal size. The specification of the latter test for $k = 2$ leads to a two sided F-test with unequal critical regions in the tails.

Finally, a distribution-free test for the k-sample problem is derived. If each x is replaced by its rank in the overall sample, the decision statistic is Max $[n_i(\bar{r}_i - n + 1/2)]$, where $\bar{r}_i$ is the mean of ranks in the ith sample, n_i its size and $n = \Sigma n_i$. The particular case for $k = 2$ leads to the Mann-Whitney test.

(E. Walter)

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ROY, J. (University of North Carolina) **5.8** (4.8)

Step-down procedure in multi-variate analysis—*In English*
Ann. Math. Statist. (1958) **29**, 1177-1187 (6 refs.)

The step-down procedure is used here to derive tests of significance and simultaneous confidence bounds, of interest in multi-variate analysis of variance and in the comparison of variance-covariance matrices. The use of the step-down procedure presupposes that the basic multi-normal variables can be ordered by the experimenter according to importance. A hypothesis concerning the multi-variate distribution is then decomposed into a number of hypotheses concerning univariate distributions. The first hypothesis concerns the marginal distribution of the first variate, the second hypothesis concerns the marginal distribution of the second variate given the first, and so on. Compound tests are developed for multivariate analysis of variance, and for the comparison of variance-covariance matrices from suitable univariate tests. In addition, simultaneous confidence bounds are found for a number of parametric functions.

It is of interest to note that the confidence bounds are frequently derived from univariate confidence bounds of the following sort: suppose the elements of $y^1 = (y_1, y_2, \ldots, y_n)$ are one-dimensional random variables, independently and normally distributed, each with variance σ^2, and with the expected value of y given by

$$\mathscr{E}(y) = A\theta + X\beta.$$

The elements of $\theta(m \times 1)$ and $\beta(q \times 1)$ are unknown parameters, and A and X are matrices of known constants, with rank $(A) = r$ and rank of $(A, X) = r+q$. Let $S^2/n-q-r$ be the usual unbiased estimate of σ^2. Suppose ϕ is a set of t estimable linearly independent linear functions of the elements of θ: $\phi = (\phi_1, \phi_2, \ldots, \phi_t)$. $\hat{\phi}$ is the best unbiased linear estimator of ϕ. Let $C = \sigma^2$ be the variance-covariance matrix of $\hat{\phi}$. If T is any subset of the natural numbers $1, 2, \ldots, t$, $T = (j_1, j_2, \ldots, j_u)$, say, and

$$T(a) = +(a_{j_1}^2 + a_{j_2}^2 + \ldots + a_{j_u}^2)^{\frac{1}{2}}$$

where $a = \phi$ or $a = \hat{\phi}$ and similarly for $a_{j_1}, a_{j_2}, \ldots$ then we can construct the following simultaneous confidence limits (with confidence coefficient, $1-\alpha$) for subsets T:

$$T(\hat{\phi}) - K \leqq T(\phi) \leqq T(\hat{\phi}) + K,$$

where $K = \lambda^{\frac{1}{2}} C_{(T)} \sqrt{\frac{s^2 t}{n-q-r} F_{(1-\alpha);\, t,\, n-q-r}}$. The quantity $C_{(T)}\sqrt{\lambda}$ is the maximum latent root of $C_{(T)}$, where $C_{(T)}$ is formed by taking the $j_1^{th}, j_2^{th}, \ldots, j_u^{th}$ rows and columns of C. The quantity $F_{(1-\alpha);\, t,\, n-q-r}$ is the $(1-\alpha)^{th}$ percentile of the F distribution with t and $n-q-r$ degrees of freedom.

(Rosedith Sitgreaves)

ROY, S. N. & ROY, J. (University of North Carolina)
A note on a class of problems in " normal " multivariate analysis of variance—*In English*
Ann. Math. Statist. (1959) **30**, 577-581 (2 refs.)

5.8 (7.3)

Given a normal p-variate population from which a random sample of n is obtained, indicated by the matrix $X(p \times n)$. The expected value of each observation on each variate is a linear function of m unknown parameters,

$$EX' = A(n \times m)\xi(m \times p).$$

The problem considered is to test the hypothesis that each of these parameters is a linear function of k other parameters, $\xi(m \times p) = B(m \times k)\eta(k \times p)$. This hypothesis is " testable " if and only if

$$\text{rank } A + \text{rank } B - \text{rank } AB \; m.$$

The test is based on the latent roots of certain sums of squares and products matrices.

(R. L. Anderson)

SILVEY, S. D. (University of Glasgow)
The Lagrangian multiplier test—*In English*
Ann. Math. Statist. (1959) **30**, 389-407 (8 refs.)

5.1 (4.1)

The material presented here is a continuation of previous work. There is the basic problem of estimating a finite dimensional parameter under the restriction that its true value be a member of a proper subset of the parameter space. The estimation problem is handled by locating a maximum of the likelihood function by the Langrange multiplier method. The use of this principle in testing the hypothesis that the true parameter lies in the prescribed subset is discussed in this paper, along with a consideration of the power of such a test.

The author details the assumptions which he imposes to define acceptable properties for this test. Under a specified set of assumptions the asymptotic joint distribution of the parameter estimator and Lagrange multiplier is obtained. Under the same assumptions, the asymptotic distribution of a statistic involving only the multiplier is obtained. In order to discuss the power of the test, attention is paid to the situation where the true parameter values does not belong to the set in which the estimate is restricted to lie. A comparison is made between the well-known likelihood ratio test, the Wald test (where acceptance of the hypothesis is based on how well the unrestricted maximum likelihood estimate satisfies the subset-defining conditions) and the Lagrangian multiplier test (where acceptance is based on how close the multiplier is to zero). It is shown that the powers of the three tests are comparable.

The test is extended to the situation where the information matrix is singular. It is also extended to situations where several random variables are being observed, but not all with the same number of observations. The author gives one example, namely, a test for homogeneity in a 2×2 contingency table. It is shown that the Lagrange multiplier statistic is the usual one used in the χ^2-test for homogeneity. More informative examples are promised in a subsequent paper.

(D. H. Shaffer)

WALSH, J. E. (The System Dev. Corp., Santa Monica, Calif.) 5.6 (4.6)
Comments on " The simple signed-rank tests "—*In English*
J. Amer. Statist. Ass. (1959) **54**, 213-224 (17 refs., 1 table)

In a fundamental contribution Tukey showed that the Wilcoxon signed-rank test is equivalent to a subclass of some tests presented by Walsh. Since Tukey's memorandum did not explicitly state that only a subclass of Walsh's tests is involved, misunderstanding seems to have arisen concerning the extent of equivalence between these two classes of tests. This paper points out that a large proportion of Walsh's results are not equivalent to Wilcoxon signed-rank tests, and that some of these non-equivalent results have useful properties.

The Walsh results that are equivalent to Wilcoxon signed-rank tests are those based on single order statistics of the entire set of $n(n-1)/2$ averages of pairs of ordered observations. The results in common represent a very small proportion of the results developed by Walsh. This proportion decreases as the number of observations increases and has a limiting value of zero.

Attributes and advantages of Walsh results that are not equivalent to Wilcoxon signed-rank tests are discussed in four sections:

1. *Computational Advantages.* Use of suitable, non-equivalent results can require less calculation, especially in determining confidence intervals.
2. *Additional Probability Levels.* The number of exact significance levels and confidence coefficients available for the equivalent results can be augmented by use of non-equivalent results.
3. *Bounded Probability Level Results.* Bounded significance level tests and confidence intervals with bounded coefficients are obtainable on the basis of some of the non-equivalent results. These tests and intervals have probability levels that are approximately valid with specified bounds for very large classes of continuous populations, symmetry is not required.
4. *Advantages for Special Situations.* In some cases large gains in efficiency can be obtained by use of suitable non-equivalent results, rather than equivalent results, if additional general nature information about the population is available. Also non-equivalent results can be adapted to test other properties such as symmetry in the tails, and to furnish a basis for non-parametric rejection of outlying observations for symmetrical populations.

A possible disadvantage of Walsh's class of results is that it is of such a great extent that several candidates are available for the same job. Thus the statistician may unconsciously pick that procedure that leads to the conclusion he favours. This type of difficulty can be avoided by choosing the procedure to be applied before the values of the observations are known.

(L. Wolins)

WIJSMAN, R. A. (University of Illinois) 5.8 (3.9)
Applications of a certain representation of the Wishart matrix—*In English*
Ann. Math. Statist. (1959) **30**, 597-601 (8 refs.)

Let X be a $p \times n$ matrix ($p \leqq n$) whose columns are independent and distributed according to a p-variate normal law with mean vector 0, and covariance matrix Σ. Any function of the Wishart matrix (XX') can be represented in terms of independent normal and χ-variates. In fact, $XX' = CTT'C'$ where $CC' = \Sigma$; T is lower triangular with independent elements T_{ij}; the diagonal elements, T_{ii} are χ-variates with $n-i+1$ degrees of freedom ($i = 1, 2, \ldots, p$) and all $T_{ij}(i>j)$ are normal deviates. If the population multiple correlation between one variate and the remaining $p-1$ is $\bar{R}$, the sample multiple correlation can be represented by $R^2/(1-R^2)$, which is a non-central F-variate with $p-1$ and $n-p+1$ degrees of freedom, with non-centrality parameter

$$T_{11}^2\bar{R}^2/(1-\bar{R}^2); \text{ for } p = 2,$$

this reduces to the non-central t-variate. T. W. Anderson's sphericity criterion Z in a bivariate population, when the hypothesis is true, can be represented by

$$Z/(1-Z) = 2T_{11}T_{22}/(T_{11}-T_{22})^2$$

which is an F-variate with $2n-2$ and 2 degrees of freedom.

(R. L. Anderson)

CLARINGBOLD, P. J. (Univ. of Sydney) **6.8** (9.2)

Orthogonal contrasts in slope ratio investigations—*In English*
Biometrics (1959) **15**, 307-322 (12 refs., 9 tables, 1 fig.)

Orthogonal contrasts discussed in this paper are of a type that will allow some special aspect of the curve of response to be studied. The problem of appropriate sets of orthogonal contrasts arises in an acute form when factor/response lines with only one parameter are encountered. Two general cases are considered in both of which the factor/response lines form a pencil of lines with a common point of intersection on the response axis. In the first case the intersection point is the origin, i.e. the response is equal to zero. In the second case, the intersection point is on the response axis above the origin, i.e. the response is greater than zero. Experimental data of this type are obtained in chemical assays where the basic parameter is optical density per unit weight, in selection experiments where advance per generation is estimated, and in slope ratio assays.

For the purpose of statistical analysis, two assumptions are made:

(*a*) The orthogonal linear model is employed throughout, experimental error being assumed independent of the treatment combinations.

(*b*) All experiments are orthogonal, of the factorial type (*N* factor).

Because of the second assumption, the contrasts for each factor may be used to generate contrasts for the whole experiment by means of the direct product (Kronecker product) of the matrices of contrasts.

After defining the direct product as *ab initio* in this paper, since the form used (a form very suitable for use in the analysis of experiments) is not given elsewhere, examples of the types of experiments mentioned earlier are given. In the first example, a selection experiment dealing with the sensitivity of albino mice to the administration of oestrogens, the methods introduced are carried out in considerable detail. In the second example, the colorimetric reaction of the natural oestrogens with sulphuric acid is studied. The matrix of contrasts suitable for examining the features of the weight/optical density lines is given, as are the initial stages of the analysis. The last example deals with a slope ratio assay in which a pair of mutually non-orthogonal contrasts is used to estimate separately the slope of each regression, line.

The paper concludes with a generalised treatment of the problem of finding and analysing orthogonal contrasts, with an example for the $(3 \times 3)+1$ and the 3×3 experiments.

(J. E. White)

DÖRING, H. & WALTER, E. (Max-Plank-Institut, Berlin) **6.2** (11.4)

On the computation of inbreeding and relationship coefficients—*In German*
Biom. Zeit. (1959) **1**, 150-161 (10 refs., 1 table, 5 figs.)

The coefficients of inbreeding and relationship were introduced in 1922 by S. Wright [*Amer. Nat.* **56**, 330-338]. The coefficient of inbreeding f_P is defined as the correlation coefficient between the uniting gametes which produced the individual (*P*). The coefficient of relationship, r_{P_x,P_y}, measures the zygotic correlation between two individuals, e.g. between *Px* and *Py*. Both correlation coefficients are based on genes identical by descent (*identiques loci*) and not on genes identical in state. The methods for computing f_P and r_{P_x,P_y} from punched cards, explained by Hazel & Lush [*J. Heredity* (1950) **41**, 301-306] are modified by the authors in order to get a " simple " formulae for the calculation of the average inbreeding coefficient $\bar{f}'$ in a given population. The same method also leads to the formulae of the average coefficient of relationship $\bar{r}$ in the same population.

Expressions as w_{XYZWi} are used; *X* denotes the individual for which the *f*-value has to be calculated, *Y* implies that we are counting the number of generations from *X* to the ancestor *W* tracing back the father-line ($Y = V$) or the mother-line ($Y = M$), *Z* is the symbol of the progeny of *W* on the path from *X* to *W*, and *i* takes care of the number of connections between *X* and *W*. w_{XYZWi} itself represents the probability, that *X* has received a gene (duplicate) from *W*. If there are *n* generations between *X* and *W*, w_{XYZWi} is equal to $(\frac{1}{2})^n$. With the notation

$$\Sigma_i w_{XYZWi} = w_{XYZW.} = \Sigma_i (\tfrac{1}{2})^{n_i}, \text{ etc.,}$$

it follows that

$$\bar{f}' = F[w_{X..W.}, w_{XY.W.}, w_{X.ZW.}, w_{XYZW.}, \text{ and cf.}],$$

and

$$\bar{r} = F[w_{...W.}, w_{X.ZW.}, w_{X..W.}, \text{ and } w_{..ZW.}],$$

where *F* means function of .., and *cf* stands for correction factor.

Henceforth it is possible to calculate the figures mentioned under (*a*), (*b*) and (*c*) [see below] with the two additional terms $w_{.Y.W.}$ and $w_{.YZW.}$. With punched cards it is easy to get the basic results which are necessary to carry out the final steps.

The numerical solutions for a simple case are given. If the average inbreeding coefficient and the average coefficient of relationship are calculated for one population, it is possible to get without much additional work (*a*) the direct relationship between any ancestor and the population, (*b*) the total and the specific amount of inbreeding caused by any ancestor, and (*c*) the average relationship *c*1: for the fathers between themselves, *c*2: for the mothers between themselves, and *c*3: for all possible father-mother pairs.

(H. L. le Roy)

HOEL, P. G. (University of California, Los Angeles) 6.1 (4.2)
Efficiency problems in polynomial estimation—*In English*
Ann. Math. Statist. (1958) **29**, 1134-1145 (11 refs., 1 table)

Given the model $\mathscr{E}(y_i) = \beta_0+\beta_1 x_i+\ldots+\beta_k x_i^k$ then a basic problem in statistics is how best to estimate the β's.

There are two aspects of this estimation problem. One is to determine the best method for using the information given by a set of n observations $y_1 \ldots y_n$. The other is to determine the best method for choosing the x values at which to take observations.

This paper gives an optimum solution based on the generalised variance for the problem of how to choose the x values in an interval for the classic regression model. A beginning is also made on the more general problem of how to choose x values for efficient polynomial estimation without the assumption that the y's are uncorrelated.

A matrix formula is given for estimating the β's: these estimators are optimum if the class of estimators is restricted to that of linear estimators. A determinant formula is given for the generalised variance of these estimators. The same formula can be obtained by assuming the y's to be multivariate normal and then finding the maximum likelihood estimates of the β's.

The author applies the criterion of generalised variance to optimise the selection of the x values for the classical regression model and goes on to investigate how best to choose the x's when the y's are not uncorrelated.

Results are obtained on the increased efficiency arising from doubling the number of equally spaced observation points:

(i) when the total interval is fixed
(ii) when the total interval is doubled.

Measures of the increased efficiency are found for the classical regression models based on a particular stationary stochastic process and a pure-birth stochastic process: the latter selected because the properties of its covariance matrix are different from those of the other models.

A table of numerical results is given. These results appear, according to the author, to indicate that in general the attempt should always be made to extend as far as possible the range over which observations are to be taken. The higher the degree of the polynomial the greater is the advantage. The results seem to indicate that if the range cannot be extended it is considerably more efficient to replicate the experiment than double the number of observations—particularly if the variables are strongly correlated.

(R. A. Fox)

KASTENBAUM, M. A. & LAMPHIEAR, D. E. (Oak Ridge Nat. Lab. and Univ. of Michigan) 6.9 (0.1)
Calculation of Chi-square to test the no three-factor interaction hypothesis—*In English*
Biometrics (1959) **15**, 107-115 (3 refs., 3 tables)

The test of the hypothesis of no three-factor interaction in an $r \times s \times t$ contingency table presents a considerable computational problem. This paper demonstrates a technique based on a procedure due to Norton [*J. Amer. Statist. Ass.* (1945) **40**, 251-258], with Newton's method of functional iteration.

The authors, after stating the null hypothesis for an $r \times s \times t$ contingency table and the multinomial distribution involved, discuss the iterative solution of the resulting $(r-1)$ $(s-1)$ $(t-1)$ third degree simultaneous equations. They then give four suggestions and one condition that should be kept in mind when using the procedure outlined in this paper. This is followed by the calculation of a statistic which is distributed approximately as chi-square with $(r-1)$ $(s-1)$ $(t-1)$ degrees of freedom and which provides a test of the null hypothesis of no three-factor interaction.

The main portion of this paper is devoted to an illustrative example based on data from an experiment performed by the Oak Ridge National Laboratory. The data provide the authors with a $2 \times 3 \times 5$ table for which eight simultaneous third degree equations in eight unknowns are stated. Accompanying this is a table for the first iteration followed by a table of the expected number of observations per cell on the null hypothesis.

The technique developed by the authors is well suited either for a desk calculator or for high-speed digital computers. The iterative solution for the example cited above was obtained by using the Oak Ridge Automatic Computer and Logical Engine. The authors mention that there is a more general programme for the Oracle which enables one to obtain the complete solution to the above problem in less than sixty seconds. This programme is adequate for tables of sizes from $2 \times 2 \times 2$ to $5 \times 5 \times 16$.

(B. J. Trawinski)

KEMPTHORNE, O. & NORDSKOG, A. W. (Iowa State College) **6.4** (4.8)
Restricted selection indices—*In English*
Biometrics (1959) **15**, 10-19 (2 refs., 4 tables)

This paper presents a derivation and examples of restricted selection indices maximising the gain in genotypic economic value when errors of estimation are ignored.

For the development of an index, it is assumed that the genotypic value (G_i) represents the breeding value which is the least squares prediction of genotypic value on the basis of a model containing only additive effects of genes, and that deviations of actual genotypic values from predicted values behave completely as random variables. Also included is the condition that the phenotypic value (P_i) shall be made up additively of two parts, a genotypic value and an environmental contribution. It is possible to permit interactions of genotype and environment provided the association is entirely at random and any such interaction then is incorporated into the environmental contribution. With these assumptions, the offspring of a mating will have a genotypic value equal to the average of the breeding values of the parents.

If, under these assumptions, selection based on a linear function (I) of the phenotypic values is made, the choice of selection index reduces to finding that linear function which correlates best with the genotypic economic value (H). The index obtained maximises gain in H when errors of estimation are ignored.

While use of I will result in best progress in H, the means of the G_i will change in either a positive or negative direction, so that a breeder may well be interested in increasing H as much as possible with a restriction that some G_i or some linear function of the G_i will not change.

The problem can then be stated as follows:

Given the P_i, determine the linear function I so that the correlation of I with H is a maximum subject to the conditions that the covariance of I and the linear function of the G_i is zero. The paper presents the mathematical solution to this problem and gives examples of its use; indicating the general effects of restriction of index. For instance, with increasing restriction, the correlation of index with economic value decreases. The weights attached to some attributes are essentially unchanged with the imposition of restrictions while others change because of the particular correlation structure.

(W. H. Beyer)

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LEECH, F. B. & HEALY, M. J. R. (Rothamsted Experiment Station, England) **6.1** (7.6)
The analysis of experiments on growth rate—*In English*
Biometrics (1959) **15**, 98-106 (4 refs., 3 tables)

This paper presents a method of analysis for experiments in which measurements on an experimental unit are made at equal time intervals. The authors discuss the comparison of the average growth rate of a treated and a control subject. It is assumed that the difference between growth curves is a straight line through the origin. By fitting first degree orthogonal polynomials to the observations for a treated and a control subject, the authors obtain two estimates of the slope of this line (the average growth rate): one from the difference in mean yields for the pair of subjects, a second from the linear components. The linear combination of these two estimates which has minimum variance is suggested as an improved estimate of this slope.

In order to apply the derived formulae, it is necessary to have estimates for the variances and covariance of the two simple estimates of the slope. These are obtained from the residual mean squares and product in a covariance analysis. By dividing the variances and covariances of the estimates by N, the formulae can be applied to N pairs of subjects. The method is extended to more than two treatments by substituting the treatment yields in place of the differences in yields for treated and control subjects to obtain two quantities y_0 and y_1. Estimates of the relative growth rate are obtained from the differences in these quantities for any pair of treatments.

The authors then consider the case where a quadratic model is necessary. The treatment effect is written as the sum of a first and a second degree polynomial function of t through the origin, and two simple estimates as well as a combined estimate are derived for each of the coefficients λ and μ of these terms. It is shown at the end of the paper that the combined estimates are the least squares estimates of these parameters. An example is provided showing how the polynomial functions ϕ are obtained by adding a suitable constant to the f values in Fisher and Yates' tables so that $\phi = 0$ when $t = 0$. The quantities needed to evaluate the estimates of λ and μ, and their variances and covariances are also given for this particular example.

The authors recommend a covariance analysis of the initial observed differences $\delta(0)$ with a function of the mean and linear components as covariate since the latter are often highly correlated with $\delta(0)$. Adjustments to the appropriate means and covariances can be made before applying the formulae given in the paper.

An example is given in which the effects of six treatments on the relative growth rates of pigs are compared.

(I. P. Monahan)

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LIENERT, G. A. (Institute of Psychology, Marburg, Germany) **6.3** (–.–)

Principle and method of multiple factor analysis, with a practical demonstration—*In German*

Biom. Zeit. (1959) **1**, 88-141 (26 refs., 27 tables, 12 figs.)

This paper is a concise and thorough introduction to the principle and method of the Thurstone multiple factor analysis. It is intended to relieve the biologist, even more so the psychologist, of the task of working through the respective literature.

The first part (about 30 pp.) of the main body of the paper is devoted to a short historical sketch and a demonstration of why Thurstone's centroid factor method should be preferred. The reasoning being that it yields results that are readily interpretable from a biological point of view. This first section also includes an illustration of the principle and method—both graphical and numerical, together with a short introduction to matrix algebra. It starts with the solution of a dummy example and works through it the other way round, discussing the relevant points, i.e. geometrical interpretation of factors, rotation and optimal rotation, simple structure, extraction of centroid factors.

The second part (about 18 pp., including a detailed calculations scheme) provides an extensive numerical example taken from psychology with extraction of four centroid factors and three orthogonal rotations, together with an interpretation of the rotated factors. The rest of the paper is concerned with typical special problems—non-orthogonal rotation, product-moment equivalent correlation coefficients, inverse factorisation (*Q*-, *P*- and *T*-techniques), analytical rotation—and a discussion of references for further reading.

(R. Wette)

LINHART, H. (Nat. Inst. Personnel Res., Johannesburg) **6.4** (5.8)

Techniques for discriminant analysis with discrete variables—*In English*

Metrika (1959) **2**, 138-149 (4 refs., 13 figs.)

Techniques are given to classify individuals into one of $\mathcal{J}$ mutually exclusive groups, when all or some discriminators consist of a number of discrete variables $k = 1, \ldots, K$; $l = 1, \ldots, L$; $m = 1, \ldots, M$; .... In the purely discrete case, the matrix (c_{ij}) for γ the costs c_{ij} (or its linear transform $c'_{ij} = a + bc_{ij}$) of misclassifying an individual belonging to group i into group j $(i, j = 1, \ldots, \mathcal{J})$ has to be stated, and the cell $k, l, m, \ldots$ is assigned to the group j^* if

$$\Sigma_i c_{ij^*} \cdot \pi_{ikl m \ldots} \leqq \Sigma_i c_{ij} \cdot \pi_{ikl m \ldots},$$

the simultaneous distribution $\pi_{ikl m \ldots}$ of $i, k, l, m, \ldots$ either being given or otherwise being estimated as $f_{ikl m \ldots}$ from a sample. If the c_{ij} are all equal (i.e. $c_{ij} = 1$ for $i \neq j$ and $c_{ii} = 0$) a cell $k, l, m, \ldots$ has to be assigned to j^* if $f_{j^* kl m \ldots}$ is the maximum frequency in the cell. The optimum (minimised) cost can only be attained if the $\pi_{ikl m \ldots}$ are known, thus the problem of when to include additional variables $n, \ldots$ in case the π's are estimated as f's cannot be solved easily; it is suggested that not more than three variables are used before more precise methods are available.

This analysis, which is easy to perform, is well suited to serve as a nonparametric discriminatory analysis on continuous variables if the pre-requisites of classical analysis are not satisfied. If some variables, e.g. $x, y, z, \ldots$, are continuous and some discrete, the usual techniques for continuous discriminatory analysis pertain, using the conditional distribution $\pi_{i \mid kl m \ldots}$ as the *a priori* probabilities of belonging to group i. Some difficulties arise if more than one continuous variable is included, since then it generally has to be assumed that the conditional joint distributions

$$f(x, y, z, \ldots \mid i, k, l, m, \ldots)$$

are multivariate normal with identical moment matrices and, for fixed i, identical means. The assumptions that can be relaxed have to be decided in each instance. In the case of only one continuous variable most restrictions can be dropped and the arithmetic of the analysis is straightforward as long as the number of cells $k, l, m, \ldots$ is not too large.

One example is given and the author acknowledges recent work of Bahadur cited by Sitgreaves in 1957 [*Biometrics* **13**, 545] which was unknown to him prior to completion of his own studies.

(R. Wette)

QUANDT, R. E. (Princeton University) **6.1 (4.3)**

The estimation of the parameters of a linear regression system obeying two separate regimes—*In English*

J. Amer. Statist. Ass. (1958) **53**, 873-880 (5 refs., 1 table, 2 figs.)

The parameters of a linear regression system may change abruptly at some point in time, so that all observations made prior to the switch obey one linear regression law while later observations obey another linear regression law. When the position in time of this " switching point " is unknown it must be estimated in order to permit estimation of the two sets of regression parameters. If the residuals are normally distributed, maximum likelihood estimates of the two sets of regression parameters may be obtained for any fixed switching point provided the number of observations under each law is large enough for non-degeneracy. By employing these estimates the likelihood of the sample may be calculated for fixed non-degenerate switching points.

Since the sample likelihood is not a continuous function of the switching point, the maximum likelihood estimate of this point is not obtainable from the derivative of the likelihood. Nor may the estimate be found by solving the usual inequality for the mode of a discretely varying function. Hence all permissible switching points must be considered, finding the likelihood (more readily its logarithm) for each point and taking as estimate that point for which the sample likelihood is a maximum. This estimate and the corresponding two sets of regression parameter estimates then comprise the desired set of maximum likelihood (and least squares) estimates of the unknown parameters of the regression system.

It is possible to test the hypothesis that no switch occurs in a given time interval against the alternative of a single switch. Such a test may be based upon the large sample chi-square approximation to the distribution of the likelihood ratio. Or the small sample variance ratio (F) test arising from the general linear hypothesis may be employed. When the position of the switching point must be estimated from the sample the latter test is not exact. The power of each test is considered and is found to depend on the position in time of the switching point. A worked example employing data from a sampling experiment is included.

(S. Krane)

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RADNER, R. (University of California, Berkeley) **6.1 (4.2)**

Minimax estimation for linear regressions—*In English*

Ann. Math. Statist. (1958) **29**, 1244-1250 (4 refs.)

In connection with the problem of estimating the coefficients of a linear regression it is usually assumed that the covariances of the observations of the dependent variables are known up to multiplication by some common positive (but unknown) number (c).

If $c<k$, and if the rest of the possible distributions of the dependent variable includes " enough " normal distributions (in a sense specified in the paper), then the minimum variance linear unbiased (Markoff) estimators of the regression coefficients are minimax among the rest of all estimators independent of k.

The author considers the problem of minimax estimation in the general linear regression framework when less is known about the covariances of the observations on the dependent variable than in the traditional restriction described above. From the minimax point of view some information is necessary otherwise the risk of every estimator is unbounded. In practice it is assumed that the covariances are bounded and the problem is then to find an expression for the bounds.

Two methods of bounding the covariances are investigated. The first is equivalent to choosing a coordinate system for the dependent variables and placing a bound on the characteristic roots of the matrix of covariances of the coordinates, e.g. placing a bound on the trace on the covariance matrix, or on its largest characteristic root. The second method consists of choosing a coordinate system, and then placing a bound of the variance of each coordinate.

The first method of bounding is shown to give the better minimax estimators, but in both cases the estimator depends upon the choice of coordinate system, which is a disadvantage if there is no " natural " coordinate system intrinsic to the regression problem being considered.

The theory developed in the paper is illustrated with four worked examples.

(R. A. Fox)

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RAO, C. (Indian Statistical Institute, Calcutta) **6.1** (5.8)

Some problems involving linear hypotheses in multivariate analysis—*In English*
Biometrika (1959) **46**, 49-58 (13 refs., 1 table)

A generalisation of the linear model, using correlated residual random variables, is taken as the basic model. The specific application is in the analysis appropriate to p-variate observations which may be represented by (p-variate) multinormal sets of random variables, with variance-covariance matrices with unknown absolute values but in known ratios one to the other. The problems discussed are :—

(i) Is the model adequate ?

and, assuming (i) to be answered affirmatively,

(ii) How can estimates of the unknown parameters be obtained ?

(iii) How can general linear hypotheses about values of these parameters be tested ? and

(iv) How can simultaneous confidence intervals for sets of linear functions of these parameters be constructed ?

The first of these is considered, for the most part, from the standpoint of curvilinear regression, and is answered by an application of Hotelling's T-distribution. The third is answered by a modification of Kolodzieczyk's approach, and the solution of (iv) uses a natural extension of the methods of Bose and Roy. The second problem is not treated explicitly but appears incidentally in the extensive analysis of a numerical example illustrating the methods described in the paper.

(N. L. Johnson)

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RÉNYI, A. (Math. Inst., Hungarian Academy of Sciences, Budapest) **6.0** (6.9)

New version of the probabilistic generalisation of the large sieve—*In English*
Acta Math. Acad. Sci. Hung. (1959) **10**, 217-226 (13 refs.)

In this paper the author deals with the different measures of dependence of two random variables and a sharper version of his previous theorem on the large sieve [A. Rényi, *Magy. Tud. Akad. Mat. Kut. Intézet. Közleményei* (1958) **3**, 199-206].

The following measures of dependence are considered:

1. The correlation coefficient:
$$R(\xi, \eta) = M(\xi\eta) - M(\xi)M(\eta)/D(\xi)D(\eta).$$
2. The correlation ratio:
$$\theta_\xi(\eta) = D\{M(\eta \mid \xi)\}/D(\eta).$$
3. The maximal correlation:
$$S(\xi, \eta) = \sup_{f, g} \mid R\{f(\xi), g(\eta)\} \mid,$$
where $f(x)$ and $g(x)$ run over all Borel-measurable functions for which $M\{f(\xi)\}$, $D\{f(\xi)\}$ and $M\{g(\eta)\}$, $D\{g(\eta)\}$ exist.
4. If
$$P_\xi(A) = P(\xi \in A),\ P_\eta(A) = P(\eta \in A)$$
and
$$P_{\xi\eta}(A) = P((\xi, \eta) \in A)$$
are measures on the Borel-sets of the real line and the plane respectively, then the expression:
$$\phi(\xi, \eta) = +\left[\int_{X*Y} \left(\frac{d^2P_{\xi\eta}}{dP_\xi dP_\eta} - 1\right)^2 dQ_{\xi\eta}\right]^{\frac{1}{2}}$$
is the mean square contingency (where $X*Y$ denotes the plane and $Q_{\xi\eta} = P_{\xi*}P_\eta$ is also a measure of the dependence).

The following results are proved:
$$\theta_\xi(\eta) = \sup_g \mid R\{\eta, g(\xi)\} \mid$$
where g runs over all Borel-measurable real functions $y = g(x)$ such that the mean and variance of $g(\xi)$ exist,
$$\mid R(\xi, \eta) \mid \leqq \min\{\theta_\xi(\eta), \theta_\eta(\xi)\} \leqq \max\{\theta_\xi(\eta), \theta_\eta(\xi)\} \leqq S(\xi, \eta) \leqq \phi(\xi, \eta).$$
$S(\xi, \eta) = 0$ if and only if ξ and η are independent.

If there is an arbitrary functional dependence between ξ and η, then $S(\xi, \eta) = 1$.

The special case of some of these results has already been reported [H. Gebelein, " Das statistische Problem der Korrelation als Variations—und Eigenwertproblem und sein Zusammenhang mit der Ausgleichrechnung," *Zeit. angew. Math. Mech.* (1941) **21**, 364-379].

The new large sieve theorem states: If
$$\mid \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} S(\xi_n, \xi_m) x_n x_m \mid \leqq C \sum_{n=1}^{\infty} x_n^2$$
where C is a positive constant and $\{x_n\}$ denotes any sequence of the real numbers such that $\sum_{n=1}^{\infty} x_n^2 < +\infty$, then $\sum_{n=1}^{\infty} \theta_{\xi n}^2(\eta) \leqq C$.

(P. Révész)

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RÉNYI, A. (Math. Inst., Hungarian Academy of Sciences, Budapest) 6.0 (6.9)
On the probabilistic generalisation of the large sieve of Linnik—*In English*
Magy. Tud. Akad. Mat. Kut. Intézet. Közleményei (1958) **3**, 199-206 (7 refs.)

In some of his previous papers the author gave the probabilistic generalisation of the large sieve of Linnik; in its original form this was a method of number-theory. In this paper a sharper version of his previous theorem is proved.

Let $\xi_1, \xi_2, \ldots$ be a sequence of discrete random variables; let z_{nk} denote the possible values of ξ_n; and let A_{nk} be the event $\xi_n = z_{nk}$. Finally, let the mean square contingency of ξ_n and ξ_m be defined by the positive square-root of the quantity

$$\phi^2(\xi_n, \xi_m) \sum_k \sum_l \{(PA_{nk}A_{ml}) - P(A_{nk})P(A_{ml})\}^2 / P(A_{nk})P(A_{ml}).$$

The following theorem is proved.

Let us suppose that

$$\left| \sum_{n \neq m} \sum \phi(\xi_n, \xi_m) x_n x_m \right| \leqq B \sum_n x_n^2,$$

where B is positive constant, $\{x_n\}$ denotes any sequence of the real numbers such that $\sum_n x_n^2 < +\infty$. Then

$$\sum_{n=1}^{\infty} \theta_{\xi_n}^2(\eta) \leqq 1 + B,$$

where

$$\theta_{\xi_n}^2(\eta) = M\{[M(\eta \mid \xi_n) - M(\eta)]^2\} / D^2(\eta)$$

and η is an arbitrary random variable.

(P. Révész)

ROBSON, D. S. (Cornell University, Ithaca, N. York) 6.1 (0.2)
A simple method for constructing orthogonal polynomials when the independent variable is unequally spaced—*In English*
Biometrics (1959) **15**, 187-191 (4 refs., 4 tables)

Curve fitting by orthogonal polynomials is often used in experimental research for its many advantages over the usual regression procedure. Orthogonal polynomials are also used to construct orthogonal contrasts among levels of a treatment factor, and several methods of construction have been developed for the case in which the levels are given at equal intervals. In this paper, a simple method is presented for the construction of orthogonal polynomials when the levels of the treatment factor are not equally spaced to facilitate the analysis of data under such conditions.

The regression equation, fitted to the data by the least squares method, is expressed in a form in which the ith term is a polynomial of degree i in the independent variable (the treatment factor in this case). These polynomials must satisfy the conditions for orthogonality and normality of functions. The ith regression coefficient of this new equation is a linear function of the treatment yields and the ith orthogonal polynomial mentioned above; these coefficients also represent the orthogonal contrasts among the different levels of the treatment factor. The polynomials can be obtained from a recursive relationship by which each polynomial is determined in terms of the preceding ones, the first of them being equal to the reciprocal of the square root of the total number of levels. Proof of the recursion formula is given, with reference to the use of basically the same procedure followed by Fisher in the case of equally spaced variables.

A 3×4 factorial experiment is presented to illustrate the construction procedure. The orthogonal polynomials representing the mean, linear and quadratic effects of the first factor are given in a table; the simplified coefficients and divisors appear in a second table. The working coefficients for the comparisons among the levels of the second factor are given in a similar fashion. The working coefficients for the twelve orthogonal contrasts among the twelve treatment combinations are calculated by forming the Kronecker product of the coefficients for each factor. Reference is made to a construction procedure, also recursive, but which requires the solution of a system of linear equations.

(Carmen A. Perez)

SOMERS, R. H. (Bureau of Applied Social Research, Columbia Univ.) **6.5** (6.2)

The rank analogue of product-moment partial correlation and regression, with application to manifold ordered contingency tables—*In English*

Biometrika (1959) **46**, 241-246 (8 refs., 2 figs.)

This paper demonstrates that a generalised partial correlation coefficient can be defined by a simple extension of Daniels' theory [*Biometrika* (1944) **33**, 129-135]. Consider n objects (numbered in arbitrary order); each pair of objects is assigned a score for its value on each of the variables. These sets of scores are the components of vectors in an n^2-dimensional space. Daniels' generalised correlation coefficient is the cosine of the angle between the vectors corresponding to a pair of variates; by suitable choices for the scores, this can be reduced to product-moment r, Spearman's rho (ρ), or Kendall's tau (τ).

A first-order partial coefficient can be regarded as the cosine of the angle between two planes, each of which contains two of the three vectors involved. It can be written concisely using a notation involving a vector product, and this suggests the generalisation to partial coefficients of any order. These vector operations are also valid when there are tied ranks.

Complete and partial regression coefficients can be defined by analogy with the product-moment case; two two-variable examples (one of which involves tied ranks) are treated in detail.

Further work is in progress. Finally there are some comments on the relative importance of "operational meaning" and "theoretical power" of such coefficients.

(C. L. Mallows)

TAUBENHEIM, J. (Heinrich-Hertz Institut, Berlin) **6.9** (6.2)

A simple measure of correlation—*In German*

Gerlands Beiträge zur Geophysik (1958) **67**, 295-303 (4 refs., 1 graph)

Assuming a bivariate normally distributed random variable (X, Y) with the medians x_m and y_m, the author defines the common quadrant measure of correlation

$$\varrho = \{P_r(X>x_m,\ Y>y_m)+P_r(X<x_m,\ Y<y_m)\} - \{P_r(X>x_m,\ Y<y_m)+P_r(X<x_m,\ Y>y_m)\}.$$

He proves the well-known relation $\rho = \sin(\varrho\,\pi/2)$, where ρ is the coefficient of correlation of the bivariate normal distribution. The author suggests the use of the correlation-characteristic (Korrelations-Kennzahl) $R = \sin(\varrho\,\pi/2)$ also in the case, when (X, Y) are not normally distributed. As the correlation-characteristic is invariant under monotone transformations $\xi(x)$ and $\eta(y)$ of x and y respectively, R will equal the coefficient of correlation of the corresponding normal bivariate distribution of the random variable $(\xi(X), \eta(Y))$ if there exist transformations such that $(\xi(X), \eta(Y))$ has a two-dimensional normal distribution. According to the author, the importance of the correlation-characteristic R for the natural sciences lies also in the fact that it does not complicate the calculation if there are values which are above the upper or below the lower limit of measurability.

A test for the hypothesis $\rho = 0$ is easily performed, as q^2N is distributed as χ^2 with 1 d.f. if the hypothesis is true. In this formula, N is the sample size and q is the estimate of ϱ obtained from a 2×2 contingency-table, using the sample-medians as the basis of classification.

Finally, the correlation-characteristic R is compared with a quadrant measure of correlation introduced by Rawer.

(J. Pfanzagl)

THEIL, H. & KLOEK, T. (Econometric Institute, Netherlands School of Economics, Rotterdam) **6.6** (4.8)
The statistics of systems of simultaneous economic relationships—*In English*
Statist. Neerlandica (1959) **13**, 65-89 (32 refs., 3 tables)

This article is concerned with the estimation of parameters of systems of simultaneous economic relationships. After some introductory sections it gives a survey of research which was done in the Econometric Institute of the Netherlands School of Economics in the period: fall 1956-spring 1958.

As examples of economic equation systems are given (i) an (extremely simple) Keynesian model, (ii) G. Tintner's model of the American meat market [*Econometrics*, New York, 1952], and (iii) L. R. Klein's small model of the American economy in the inter-war period [*Economic Fluctuations in the United States*, 1921-1941, New York, 1950]. Further, some basic concepts are introduced, such as exogenous and endogenous variables; the completeness of a system; structural equations *v.* reduced forms; static *v.* dynamic systems and jointly dependent *v.* predetermined variables.

The usual assumptions which underly the estimation procedure are given, such as completeness of the system, existence of the reduced form and certain properties (zero mean, zero lagged covariances and constant current variances and covariances) of the distribution of the disturbances.

A description is given of the two-stage least-squares estimator on which the major part of the rest of the article is based. The conditions of identification are discussed in the same terminology and symbols.

The two-stage least-squares estimator can be generalised in more than one way, namely analogous to Aitken's generalised least-squares method [*Pro. Roy. Soc. Edinb.* (1934-1935) **55**, 42-48] and as the so-called general *k*-class estimator. It can be shown that the classical least-squares, the two-stage least-squares and the limited-information maximum-likelihood estimators are all members of this family of *k*-class estimators. Some other new members (due to A. L. Nagar) of this family are dealt with, viz., an estimator which is unbiased to the order $1/T$ (where T is the number of observations in the sample), and one which is " best " in the sense that the determinant value of its moment matrix is a minimum (to the order of $1/T^2$).

A preliminary solution by means of principal-component methods is given for the problems which arise when the number of predetermined variables is not small in comparison with the number of observations in the sample used. It is applied to Klein's model. [A more extensive survey will be given by Kloek and Mennes in *Econometrica* (1960) **28**, 1-17.] Another topic is the iterative estimation method which takes in account all *a priori* restrictions of the whole system. (Applications to Tintner's and Klein's models.) The distribution of the marginal propensity to consume in the simple Keynesian model mentioned above was considered in somewhat greater detail. Finally, Hooper's trace correlation coefficient [*Econometrica* (1959) **27**, 245-256] is discussed. It is based on canonical correlations and measures the extent to which the variation in all jointly dependent variables is accounted for by the set of all predetermined variables. It was computed for Tintner's model.

(T. Kloek & H. Theil)

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TURNER, M. E. & STEVENS, C. D. (University of Cincinnati) **6.6** (6.1)
The regression analysis of causal paths—*In English*
Biometrics (1959) **15**, 236-258 (23 refs., 2 tables, 1 fig.)

In this expository paper, the path analysis of causal networks originated by Sewall Wright is developed through the use of linear structural equations containing no measurement error, and with only independent and additive error components in the response variable. Since any causal network without closed loops or cycles can be composed of ordinary multiple regression, simultaneous regression, and chain regression, examples of these three relations are considered and estimates found for their parameters. Regression equations are then formulated from a path diagram of a network embodying these three basic models, and estimates of their original intercept and " path coefficient " parameters follow in terms of the partial regression coefficients. However, this algebra may be obviated by two rules:

Rule 1. A *total path regression* between a primary factor and an effect is the sum of the compound path regressions connecting the primary factor and effect, where the " compound path coefficient " is the product of the elementary path coefficients along any one path from a particular primary factor to a particular effect.

Rule 2. The *total intercept* for a particular effect η_i is the sum of the particular intercepts α_i and the products of all intercepts of effects determining η_i, by the elementary path regressions connecting the determining effect and η_i.

If each parameter can be uniquely expressed as a function of the simple and partial regression coefficients, the system is said to be *just identified*. *Over-identification* obtains when the model contains more regression coefficients than path coefficients, while the converse is known as *under-identification*. In the former situation, maximum likelihood generally gives rise to iterative solutions for the parameters, while additional information (restrictions on relative magnitudes, etc.) is required for a determinate solution in the latter case. Classical factor analysis is an extreme example of under-identified systems, where the path coefficients emerge as factor loadings.

The experiment of Haldane and Priestly provides an example of feedback in a causal network. The equilibrium of this model may be expressed as a function of the path coefficients, which are estimated with the aid of side conditions imposed by the mechanism of respiration. The path coefficients computed from the experimental data imply a negative feedback, and hence a stable system—as originally concluded by Haldane and Priestly. Since a common variance cannot be assumed for the two regression coefficients entering into this feedback model, it is necessary to employ Satterthwaite's approximation distribution of linear combinations of chi-squares. Significance levels for these path coefficients are given from the Smith-Satterthwaite, Welch, and Behrens-Fisher solutions to that problem.

Estimators involved in path analysis of non-linear relations are usually quite complicated. Model equations are developed for simple chain regression of two exponential processes, wherein estimates follow from an iterative scheme.

(D. F. Morrison)

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WAYMAN, P. A. (Radcliffe Observatory, Pretoria) 6.1 (4.3)

A least-squares solution for a linear relation between two observed quantities—*In English Nature* (1959) **184**, 77-78 (5 refs., 1 fig.)

The problem considered is that of estimating the coefficients of a linear functional relationship between two quantities which are contaminated by errors of observation; these errors are supposed to be uncorrelated and to have known variances (possibly different for each observation).

There is some discussion and criticism of related results obtained by Hertzsprung (1922), Seares (1944, 1945), and Trumpler and Weaver (1953). It is remarked that the solution given by Jeffreys does not reduce to Hertzsprung's (standard) solution when the two error variances are equal.

The present approach is claimed to be similar to that of Jeffreys, uniform prior probability for the two coefficients being assumed. The method and results are equivalent to those given by ordinary least squares [see e.g. Lindley's review article: *J. R. Statist. Soc.* B (1947) **9**, 218-244]. The two simultaneous equations to be solved are given explicitly; their solution requires numerical iteration unless the error variances are constant.

(C. L. Mallows)

BIGGERS, J. D. (Dept. of Physiology, Royal Veterinary College, London) 7.4 (9.2)
The estimation of missing and mixed-up observations in several experimental designs—*In English*
Biometrika (1959) **46**, 91-106 (23 refs., 9 tables)

The simple analysis of most experimental designs depends upon their fully balanced arrangement. If some observations are lost the technique of inserting values which minimise the error sum of squares has been used to overcome the difficulties in the statistical analysis. In this paper the problem of calculating these missing values is considered for five designs, namely (1) randomised block, (2) randomised block with replications within units, (3) cross-over, (4) latin square and higher squares, (5) split plot. The general solution in the case of p missing observations is obtained from the solution of p simultaneous equations which are expressed in matrix form. The missing values can then be obtained by inverting a matrix.

In describing an observation in a design, the classifications involved may be indicated by the number of subscripts. Pairs of observations may be called zero order, first order, etc. associates if they have no subscripts, one subscript, etc. in common. Rules for writing down the matrices, depending on the types of associates among the missing values, are given for each of the designs above: examples are worked through for these designs.

Previous workers have given specialised results, and have usually recommended iterative methods when more than one missing observation exists. Until very recently they have not considered the simultaneous equations explicitly. The main advantage of the present procedure is that these equations can be examined, and no iteration is needed.

Sometimes the identity of observations may be lost so that we know only their total. These are known as mixed-up observations. In the last section the general case where both missing and mixed-up observations occur is related to the previous sections by minimising the error sum of squares subject to conditional equations, and the matrix equation is given with a worked example.

(N. W. Please)

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CURNOW, R. N. (Agric. Res. Council, Univ. of Aberdeen) 7.3 (9.1)
The analysis of a two phase experiment—*In English*
Biometrics (1959) **15**, 60-73 (4 refs., 6 tables)

The author re-examines a paper by McIntyre [*Biometrics* (1955) **11**, 324-334] concerning the design and analysis of two phase experiments. These are experiments in which the effects of the various treatments on the experimental units cannot be assessed directly and a further experiment is needed to estimate them. Generally the analysis of these experiments is straightforward and follows the pattern of the first phase design. The second phase of the experiment is analysed only if the efficiency of this phase is of interest. However, McIntyre presented also the following complex two phase experiment in which the best method of analysis is not at all obvious.

The experiment was designed to compare the effects of four light treatments on the synthesis of mosaic virus in tobacco leaves. Eight tobacco plants were divided arbitrarily into two sets of four and the light treatments applied to each set in a Latin square arrangement with plants as columns and leaf positions on plants as rows. The effects of these light treatments were estimated in the second phase by injecting sap from each of the first phase leaves into half-leaves of certain assay plants. The number of lesions which appeared on these half-leaves was counted. The design was such that the specimens of sap from the four leaves of the first plant of the first Latin square were the Latin letter treatments, and the specimens of sap from the first plant of the second Latin square were the Greek letter treatments in a 4×4 Graeco-Latin square formed by the half-leaves of four assay plants with, as in the first phase, plants as columns and leaf positions on plants as rows. Three further Graeco-Latin squares were used to assay the sap from the other six first phase plants.

The method of analysis is too involved to describe here but starts from separate analyses of variance performed on the sums and differences of the number of lesions on the two half-leaves in each cell of the Graeco-Latin squares. Two sets of estimates of the treatment effects result, corresponding to the inter- and intra-block estimates that arise in the analysis of an incomplete block design. These two sets of estimates are of different accuracies and a method is given for combining them by a suitably weighted mean. McIntyre's somewhat simpler method of analysis is equivalent to taking the unweighted mean and generally, but not in the present example, leads to less efficient estimates.

(H. A. David)

WALLACE, D. L. (University of Chicago)

Simplified Beta-approximations to the Kruskal-Wallis H test—*In English*

J. Amer. Statist. Ass. (1959) **54**, 225-230 (3 refs., 2 tables)

Use of the H statistic arises when the observations in a one-way analysis of variance table, with C samples of $n_1, n_2, \ldots, n_c$ observations respectively have been ranked $1, 2, \ldots, N = \Sigma n_i$. A Beta-approximation to the distribution of $B_1 = H/M$, with M the maximum possible value of H, was given by Kruskal and Wallis, but the use of M has been seen to cause a tendency for under-estimation of the significance level for extreme results.

The author proposes performing an ordinary one-way analysis of variance on the ranks, then the usual analysis of variance statistic $B_2 =$ between sum of squares divided by the total sum of squares $= H/N-1$. Three Beta-approximations to the distribution of B_2 are proposed. The first is one commonly used to approximate permutation test distributions in the analysis of variance. This is obtained, in general, by modifying the degrees of freedom of the fitted Beta-distribution so that its mean and variance agree with the exact mean and variance of B_2. The factor d needed for this modification is given and it is seen that the B_1 approximation is exactly the same as the B_2 approximation with $N-1$ replaced throughout by M. The B_2 approximation seems to have slight advantages in accuracy, simplicity, and in relation to the ordinary analysis of variance.

However, both B_1 and B_2 have the disadvantage of computational difficulty in getting the degrees of freedom. The other two approximations to B_2 eliminate this and are sometimes appropriate. The second approximation is obtained by calculating d as if all sample sizes were equal to the average sample size. The third approximation puts $d = 1$, which is just the ordinary normal theory analysis of variance on the ranked data. This latter approximation, however, introduces substantial errors for small samples. A table is given which bounds the changes in significance levels from using the second or third approximation to B_2 instead of the first, thus providing a guide as to when use of the simpler approximation is appropriate.

The exact distribution of H is available for three samples of size up to five. A table of numerical comparisons of the approximate significance levels with the exact level is given for three samples of sizes (3, 3, 3), (3, 4, 5), and (5, 5, 5). The Paulson approximation to F was used to obtain sufficient accuracy on the items involving fractional degrees of freedom.

(T. N. Throckmorton)

AGGARWAL, O. P. (Purdue University, Indiana, U.S.A.) **8.7** (4.9)

Bayes' and minimax procedures in sampling from finite and infinite populations, I—*In English*

Ann. Math. Statist. (1959) **30**, 206-218 (8 refs.)

Simple random sampling without replacement from a finite population and stratified sampling from both an infinite and a finite population are considered in terms of loss and risk functions. The loss function is defined as the sum of two parts, the first component proportional to the square of the error of the estimate and the second component proportional to the cost of obtaining the sample. The discussion is confined to non-sequential and non-randomised estimates.

In estimating the mean of a finite population, using simple random sampling without replacement, it is shown that the sample mean is a minimax estimate. This is done by considering an infinite sequence of *a priori* distributions for the population mean, finding the Bayes' estimate and associated Bayes' risk for each of these distributions, finding the limiting value of the Bayes' risk, and then showing that the risk associated with the sample mean is always less than this limiting value.

The next problem considered is that of estimating the function $U = \sum_{i=1}^{k} a_i\mu_i$, where a_i are real non-zero numbers such that $\Sigma a_i = 1$, and where there are k infinite populations with μ_i the unknown means and with σ_i^2 the known upper bounds for the variances. First fixing n_i, the size of the sample from the ith stratum, and assuming that the k populations are normally distributed, it is shown in much the same way as in the previous section that $\Sigma a_i\bar{X}_i$ is a minimax estimate for $\Sigma a_i\mu_i$, for fixed n_i. The assumption of normality is then removed, and the same minimax estimate is obtained assuming only that the distribution in the ith stratum has an unknown mean μ_i, and that the sample mean $\bar{X}_i$, for any sample size n_i, has a variance not exceeding σ_i^2/n_i, a known positive number. Next the minimax strategy is found to consist of choosing the n_i equal to the integer nearest to $\sqrt{(a_i^2\sigma_i^2/c_i)}+\frac{1}{4}$, where c_i is the sampling cost per observation in the ith stratum.

Finally a finite population is considered, which consists of k strata of size N_i, with unknown mean μ_i, and with σ_i^2 the known upper bounds of the variances. It is shown that a minimax estimate for $\Sigma a_i\mu_i$ for given sample size n_i is $\Sigma a_i\bar{x}_i$ where $\bar{x}_i$ is the sample mean for the ith stratum. The minimax strategy for choosing n_i is found to consist of choosing n_i to be the integer nearest to $\sqrt{(a_i^2\sigma_i^2/c_i)}+\frac{1}{4}$ and $\leqq N_i$. In particular, for estimating $\Sigma N_i\mu_i/N$, where $N = \Sigma N_i$, n_i is chosen to be the integer nearest to $\sqrt{(N_i^2\sigma_i^2/N^2c_i^2)}+\frac{1}{4}$ and $\leqq N_i$.

(Olive J. Dunn)

DALENIUS, T. & HODGES, J. L., Jr. (Stockholms Högskola and Univ. of California, Berkeley) **8.1** (–.–)

The choice of stratification points—*In English*

Skand. Aktuartidskr. (1957) **40**, 198-203 (8 refs.)

Consider a univariate population with finite variance, distributed over the interval (a, b) according to the density f. The population is stratified into L strata by stratification points x_h. The Tschuprow-Neyman theorem asserts that the variance of the estimate $\bar{x} = \Sigma W_h\bar{x}_h$ where W_h is the relative size of the hth stratum, is minimised when the total sample is allocated between the strata in proportion to $W_h\sigma_h$. It has been shown by Dalenius that for this allocation, the best choice of stratification points is given by the set x_h satisfying

$$\sigma_h^2+(\mu_h-x_h)^2/\sigma_h = \sigma_{h+1}^2+(\mu_{h+1}-x_h)^2/\sigma_{h+1}$$

where μ_h is the expectation and σ_h^2 the variance in the hth stratum.

In the present paper, the authors derive an approximate solution to the above equation. This approximation is arrived at by introducing a new variable y defined by

$$y = G(x) = \int_a^x \sqrt{f(u)}\,du$$

and applying constant-width stratification of y; that is stratification points x_h' are defined by

$$G(x_h') = hK/L$$

where $K = G(b)$ and L is the number of strata.

(T. Dalenius)

DALENIUS, T. & HODGES, J. L., Jr. (Univ. Stockholm and Univ. California, Berkeley) 8.1 (0.1)
Minimum variance stratification—*In English*
J. Amer. Statist. Ass. (1959) **54**, 88-101 (10 refs., 8 tables)

This paper starts by reviewing some earlier results due to Dalenius. He has considered a population represented by a density $f(x)$ with mean μ, and has defined the set $[x_h]$ of points of stratification by cutting the range x_0, x_L of the estimation variable x into a monotone increasing sequence of L terms. The mean μ is estimated by the weighted mean $\bar{x}$; it is common knowledge that the variance of $\bar{x}$ is minimum when using optimum allocation. This variance is a function of the points of stratification x_h. Dalenius found previously that the set $[x_h]$ satisfying the relation

(1) $$\sigma_h^2+(x_h-\mu_h)^2/\sigma_h = \sigma_{h+1}^2+(x_h-\mu_{h+1})^2/\sigma_{h+1}$$

corresponds to minimum variance stratification. This solution presents computational difficulties, although for special cases some procedures are mentioned. The purpose of this paper is to provide approximation to the exact solution. A crucial point is, that the population is represented by a density $f(x)$. The authors introduce as a first approximation the transformation $y(u)$ equal to the integral on $(-\infty, u)$ of the square root of the population density. When $u\to\infty$, $y(u)\to H$, an upper bound.

The roots of the equation

(2) $$y(u) = (h/L)H, h = 1, 2 \ldots$$

are taken as first approximations, for large L, to the points of $[x_h]$. A heuristic justification of this is presented and a reference to a rigorous proof by the same authors is given. For convenience in numerical calculations, the pth moment of the population density is defined as $I_p(u)$ on $(-\infty, u)$ and it is applied to express the conditional means and variances of two consecutive strata. The exact expression (1) is given the form $A_h-B_h = \Delta_h = 0$, where A_h and B_h are functions of $I_p(u)$. The set $[x'_h]$ found as solution of (2) is adjusted (second approximation) into $[x''_h]$ and checked with $\Delta_h = 0$. Thus an iteration method is produced. The second approximation is illustrated by considering a rectangular distribution $f(x) = 1$, $0\leqq x\leqq b$. For $L = 2$ and $L = 3$ the solution is spelled out. For any number L of strata, Δ''_h is given as a function of $[x'_h]$, $[x''_h]$ and Δ'_h. The matrix of the partial derivatives of Δ'_h with respect to x'_h is given as M, an $(L-1)\times(L-1)$ continuant matrix whose inverse provides the solution. The authors point out that the procedure holds for some population densities other than the rectangular. Applications follow to exponential and other density functions. Comparisons of different rules for stratification, mentioned in the introduction to the paper, are presented, as well as a comparison of different computing techniques. It is reported that the new technique saves considerable computing time.

(J. N. de Pascual)

EKMAN, G. (Inst. Math. Statist., University of Stockholm) 8.1 (4.9)
An approximation useful in univariate stratification—*In English*
Ann. Math. Statist. (1959) **30**, 219-229 (5 refs., 1 table)

A population with known density $f(x)$ is assumed; the population is to be divided into n strata on the basis of the variable x (n fixed), but the points of division arbitrary. Dalenius has given a formula in [*Skand. Aktuar.* (1950) **33**, 203-213] for the division points which minimise the variance of a stratified random sample estimate of the population mean, $\bar{x}$. This formula involves the strata means and variances which in turn depend on the division points so that it can only be solved (in general) iteratively; hence, for large n it will be very difficult to use.

In the present paper a simpler approximate formula is given for these division points. This involves only the stratum size and a constant depending on n so that while an iterative solution is still necessary, it is far simpler to use than the equation for the exact minimum variance division points. The approximate formula is asymptotically correct as n, the number of strata, tends to infinity but numerical examples are given to show that even for n as small as two the approximation is adequate. No discussion is given of what happens when the division is made on the basis of estimates of $f(x)$, as it must be in any practical problem.

(D. G. Chapman)

FISHER, W. D. (Kansas State College) 8.1 (11.5)

On grouping for maximum homogeneity—*In English*

J. Amer. Statist. Ass. (1958) **53**, 789-798 (11 refs., 1 table, 2 figs.)

Given a set of K elements, each element having assigned to it a weight, w_i, and a numerical measure, a_i, and given a positive integer $G<K$, the problem is to obtain a systematic and practical procedure for grouping the K elements into G mutually exclusive and exhaustive subsets such that the sum of squares within groups, $D = \sum_{i=1}^{K} w_i(a_i-\bar{a}_i)^2$ is minimised, where $\bar{a}_i$ denotes the weighted arithmetic mean of those a's that are assigned to the subset to which element it is assigned.

Two sub-classes of the grouping problem are distinguished: (1) the unrestricted problem, where no restrictions or side conditions are imposed on the partitions allowed; and (2) the restricted problem, where such conditions are imposed *a priori* on the basis of previous knowledge, theory or for convenience.

To find the optimal grouping for both the unrestricted and restricted problems it is sufficient to compute the D values for each of the $\binom{K-1}{G-1}$ possible contiguous partitions of K elements into G groups and then select the one with minimum D. In the unrestricted problem the K elements are ordered according to a criterion under investigation, whereas in the restricted problem the elements are ordered according to *a priori* knowledge. For larger values of K and G the number of combinations becomes so large that consideration of all possible groupings becomes impractical even for the fastest digital computers. However it is possible to reduce the computations substantially by the use of suboptimisation procedures.

A programme for the "Illiac" automatic digital computer at the University of Illinois is available for solving the unrestricted grouping problem, or the restricted problem when the elements are completely ordered *a priori* with capacity $K\leqq 200$, $G\leqq 10$. The programme solves the problem to the extent of identifying one optimal partition and its associated D value for $G = 1, 2, \ldots, \bar{G}$ where $\bar{G}$ is specified as the maximum number of subsets to be considered, and cannot exceed ten.

(S. Addelman)

JACKSON, J. E., FREUND, R. A. & HOWE, W. G. (Virginia Poly. Inst. and Eastman Kodak) 8.9 (5.1)

Errors associated with process adjustments—*In English*

Virginia J. Sci. (1959) **10**, 3-26 (1 ref., 1 table, 6 figs.)

In addition to being concerned with risks associated with errors of Type I and Type II many industrial people must consider the errors of over and under correction when making process adjustments. After determining that a process is "out of control", it is necessary to estimate the actual deviation from standard in order to make the required correction. The determination of the probability of these errors is accomplished by decomposing the power curve associated with the process control procedure into:

1. The probability of undercorrecting.
2. The probability of overcorrecting but still being closer to standard than before.
3. The probability of overcorrecting and being farther from standard than before.
4. The probability of adjusting in the wrong direction.

Examples are presented of the effects of four different operational procedures. In the first two, control limits of σ, 2σ and 3σ (from the standard) were considered, where σ was the standard error of the sample average, $\bar{X}$.

(*a*) Adjust by the full amount of the difference between the standard and $\bar{X}$, when $\bar{X}$ is outside the control limits.

(*b*) Adjust by only one-half this difference when $\bar{X}$ falls between the control limit and twice the control limit; full adjustment when $\bar{X}$ falls outside the latter.

(*c*) Adjust by one-third this difference when $\bar{X}$ falls between σ and 2σ, by two-thirds this difference when $\bar{X}$ falls between 2σ and 3σ and by the full difference when $\bar{X}$ exceeds 3σ.

(*d*) Use of a two-sample procedure when analytical and product variability are both of importance.

Special attention is paid to the problem of selecting control limits which would minimise the probabilities of the adjustments producing worse results than if no adjustment were made.

(B. Harshbarger)

KLEGA, V. (Heat Eng. Res. Inst., Prague) 8.9 (2.5)

Statistical quality control of out-of-roundness of machined parts—*In English*

Aplik. Mat. (1959) **4**, 109-125 (11 refs., 5 tables)

In this paper a statistical method is presented for control of deviations from roundness of machined parts. The deviation from roundness of a cylindrical part is defined as the difference between its maximum and minimum diameter. It is supposed that the deviations are subject to Weibull's distribution. This means that a suitable power of the deviation is exponentially distributed with mean s: this hypothesis was found adequate in practical cases.

In the statistical model, the exponent of the power is supposed to know and the decision between the hypothesis that s is equal to a given value r and the alternative that s exceeds r is to be made. When we use the Neymann-Pearson principle, we see that the most powerful test of the hypothesis is based on the mean of powers of deviations. Such a test is too complicated for practice, so the author proposes to find a test using the range between the jth value from the top and ith value from below. The hypothesis is rejected with great values of the range. After a study of the asymptotic power, it is found that the maximum power is attained by the test based only on the jth value from the top, where j is approximately $0{\cdot}203n$, assuming n is the size of the controlled sample.

For purposes of statistical quality control the parameter value r of the null-hypothesis may be either estimated from preceding observations or calculated from the mean percentage of defectives tolerated. The product is classified as defective and its deviations of roundness exceeds a given value. With regard to the simple form of the distribution of the deviation, the value of r is easily obtained. The author gives tables of critical points of the distribution of the first and second value from the top for the sample-sizes between 3 and 10. So, if it is presumed that author's asymptotic expression of optimal j as $0{\cdot}203$ has its justification also for small samples, these critical points are useful for tests based on samples of 5 and 10.

Critical points for other values of j and n can be found from a table of the Incomplete Beta-Function by means of the well-known expression of the distribution of the jth value in the sample. Tables of some other auxiliary constants are also presented.

The author concludes by discussing the asymptotic efficiency relative to the most powerful test discussed at the beginning of tests based on the jth observation from the top and the ith observation from below, when j is the pth portion of the sample size and i is the qth portion of it. The efficiency as the function of p and q is tabulated. The relative efficiency of the test presented by the author is $0{\cdot}647$.

The derivations are based on a theorem proving the asymptotic normality of the distribution of the difference between the values of the order statistic from the sample of exponentially distributed variables.

(P. Mandl)

SOBOL, M. G. (Survey Research Center, Michigan University) 8.5 (8.3)

Panel mortality and panel bias—*In English*

J. Amer. Statist. Assoc. (1959) **54**, 52-68 (6 refs., 16 tables)

This paper presents the results of a panel survey which was specifically designed to permit the investigation of panel bias. A series of five interviews was held over a three year interval; an attempt was made to follow families who moved during the survey period. Economic attitude formation and change, plans to purchase automobiles and durables and the fulfilment of these plans constituted the content of this survey.

Extensive cross tabulations of panel losses—movers not followed, refusals, unavailables versus several demographic characteristics of the sample such as sex, occupation, education, age of the family head, interest in the subject matter, degree of urbanisation, are presented. Panel members (that is, respondents who replied to all five interviews) are composed statistically with panel losses on the basis of this information. A significant tendency to withdraw from the first interview-group was observed for renters, low income families and persons not interested in either the survey content or economic development in general. However, the demographic structure of the panel differs little from the original sample. This is primarily because of the compensating variations among the types of panel losses and the retention of a good proportion of the movers. Had the movers not been followed, a biased distribution among the panel members would have been obtained together with a reduction of more than 50 per cent. in sample size.

Simultaneous independent random samples of families were selected at the times of the second, third and fourth interviews and identical questions were asked. This scheme permitted significance comparisons between random samples and the panel at three different time periods. The panel was significantly more optimistic, but income differences between the panel and random sample may have accounted for this. The panel contained a larger proportion of high income families who would hence be more optimistic in any case. In general, when the effects of income distribution were eliminated, attitude differences were small. A comparison between the first interview responses of families who subsequently dropped from the survey and the panel showed but little difference.

The author concludes by offering the suggestion that the low-income units in a panel survey of an economic or financial attitude nature should be over-sampled since there is tendency for such units to withdraw.

(J. Graham)

TODA, H. (University of Tokyo) **8.9** (10.1)

Band-score control charts (I)—*In English*

Rep. Statist. Appl. Res. (1958) **5**, 54-58 (6 refs., 2 tables, 2 figs.)

It is well known that quality-control people should be on guard for runs in control-chart data even though all points fall within control limits. In this paper the author evaluates the error of the first kind of control charts when one takes action by runs above or below average, as well as by the three sigma control limits, using the sum of scores given to the points on the chart depending on which of the bands they fall in. The rules of action are as follows:

Rule 1. To each point on the chart, give a score depending on which of the bands it falls in, and compute the probability that a sample point be given a specified score.

Rule 2. Plotting the data on the chart, take the cumulative sum of scores.

Rule 3. If a point falls on the different side of the central line from the preceding one, renew the cumulative sum of scores at that point.

Rule 4. If the cumulative sum of scores becomes k or more, or if it becomes k' or less, consider that the chart shows lack of control, and take action.

Rule 5. Taking action, reduce cumulative sum of scores to zero and start a new series of the cumulative sums.

To find the error of the first kind, a sequence of the sum of scores given by the rules above is regarded as a sample from ergodic Markoff chains and Feller's theorem on the ergodic property of a periodic Markoff chain is applied.

Examples are given when the population distribution is normal.

(M. Siotani)

ZUBRZYCKI, S. (Math. Inst., Polish Acad. Sciences, Wroclaw) **8.1** (8.6)

Remarks on random, stratified and systematic sampling in a plane—*In English*

Colloq. Math. (1958) **6**, 251-264 (11 refs.)

Using the problem of estimating parameters of geological deposits as the starting point of his considerations, the author studies the relative accuracy of three sampling methods in a plane. The problem under consideration can be formulated in mathematical terms as follows. Let $y(p)$ be a plane stochastic process depending on a point p in a plane. It is assumed that this process is stationary and continuous. The mean value η of the process $y(p)$ in a given domain D is to be estimated on a basis of values of this process, observed in n points, chosen in D. It is supposed that D is a sum of n parts which are disjoint and congruent by translation. The average of the n observed values is used as the estimator of η and three ways of choosing the n points are considered, which are called random, stratified and systematic sampling. Random sampling consists of choosing the n points independently with uniform probability distribution over D. Stratified sampling consists in choosing independently one point in every one of the n parts of D, with uniform probability distribution over each of these parts. Systematic sampling consists in choosing one point in the first part of D, with uniform probability distribution over it, and in taking as remaining points the points of the other parts which correspond to the first chosen point by translations establishing the congruence of the parts.

It is stated, that estimation error of stratified sampling, as measured by the expected value of the square of the difference between η and the average of observed values, is always not greater than that of random sampling. Conditions are given for the estimation error of systematic sampling to be smaller than that of stratified sampling.

Finally the author considers the isotropic process with an exponential correlation function. It is found that for such processes the estimation error of systematic sampling can be greater than that of stratified one even if the parts of D are circles. It is shown, however, that, for any shape of the parts of D, this can not happen if these parts are large enough as compared with the coefficient c in the correlation function of the form $f(d) = \exp(-cd)$.

(S. Zubrzycki)

BOSE, R. C. & MESNER, D. M. (North Carolina and Michigan State Univ.) **9.1** (0.6)
On linear associative algebras corresponding to association schemes of partially balanced designs—*In English*
Ann. Math. Statist. (1959) **30**, 21-38 (19 refs.)

Partially balanced incomplete block designs were introduced in experimental studies by Bose and Nair [*Sankhyā* (1939) **4**, 337-372], and have recently come into fairly general practical use. The concept of the association scheme though inherent in Bose and Nair's definition was explicitly introduced by Bose and Shimamoto [*J. Amer. Statist. Ass.* (1952) **47**, 151-184] as an aid to classification and analysis. The present paper discusses the mathematical structure of association schemes, defined by the three following properties:

(*a*) Any two treatments are either 1st or 2nd, ..., or *m*th associates, the relation of association being symmetrical.

(*b*) Each treatment has a fixed number n_i of *i*th associates independent of the treatment.

(*c*) If any two treatments are *i*th associates, then the number of treatments which are *j*th associates of the first and *k*th associates of the second is p_{jk}^i and is independent of the two treatments.

Each treatment can be considered its own *O*th associate. If there are v objects then the association matrix B_i is defined as a symmetrical $v \times v$ matrix such that the element in the αth row and βth column is 1 or 0 according as the αth and βth treatments are or are not *i*th associates. It is shown that the product of any two of the association matrices $B_0, B_1, \ldots, B_m$ can be expressed as a linear function of the $m+1$ association matrices, the coefficients being the parameters p_{jk}^i, so that the matrices form a linear associative algebra. The product is commutative, and one can define $m \times m$ matrices $P_0, P_1, \ldots, P_m$ in terms of the parameters p_{jk}^i, the algebra formed by which is isomorphic with the algebra of association matrices. From this fact all the known identities between the parameters and some new ones are deduced. The multiplicities of the characteristic roots of any linear function of the B's are studied. In particular, they are shown to be independent of the coefficients, thus generalising an earlier result by Connor and Clatworthy [*Ann. Math. Statist.* (1954) **25**, 100-112]. The results of this paper would be useful in proving the non-existence of certain partially balanced incomplete block designs, and in simplifying the derivation of the normal equations for estimating treatment effects. Tables of partially balanced incomplete block designs with two associate classes, giving the designs and explaining and illustrating the analysis have been published by the Institute of Statistics, University of North Carolina [*Tech. Bull.* No. 107 (1954)].

(R. C. Bose)

BECHHOFER, R. E., ELMAGHRABY, S., & MORSE, N. (Cornell University) **9.4** (1.8)
A single-sample multiple-decision procedure for selecting the multinomial event which has the highest probability—*In English*
Ann. Math. Statist. (1959) **30**, 102-119 (9 refs., 8 tables)

Given an experiment whose outcome is one of k possible events, the problem is to decide, on the basis of N observations, which event has the highest probability of occurrence.

More specifically, a simple procedure is set up as follows: after the experiment is performed N times, the event of highest probability is selected as that event which occurred most often. (In case of ties for first place, the tied events are assigned equal probabilities and then one is selected at random). The success of this procedure depends, not only on N, but also on the configuration of the probabilities, $p_1, p_2, \ldots, p_k$, of the k events. The authors select, as a measure of this configuration, the quantity θ, which is the ratio of the largest to the second largest of the p's. The problem then is, given some θ^* and P^*, to find the smallest N such that the probability of a correct selection is at least P^*, regardless of the configuration of the p's, provided only that $\theta \geqq \theta^*$.

The problem is solved in terms of the least favourable configuration of the p's for a given θ, which is that in which all p's are equal except the largest.

In Section 5, an exact solution is given, leading to Tables A-2, A-3, and A-4. These three tables give the probability of a correct selection in the least favourable configuration for various N (1 to 30), θ^* (1·02 to 10), and for $k = 2$, 3 and 4, respectively.

Larger values of k and N lead to an excessive amount of computation, and the remainder of the paper is devoted to the derivation of approximate solutions. In Section 6.1, a normal approximation is derived; this is abandoned, as the tables necessary for its use do not exist. In Sections 6.2.1. and 6.2.2. an approximation is derived for which the appropriate tables do exist. This approximation is evaluated in Section 6.2.3 (Tables B-2, B-3 and B-4) by comparing its results with exact results in a number of cases for $k = 2$, 3 and 4. The conclusion is that " the approximation is excellent even if N is only moderately large " (10 or more); it " breaks down if θ^* is large and k is small ".

Section 6.2.4 describes some of the existing tables that can be used with the approximation. Section 7 shows how to choose N to meet the given requirements. A numerical example is provided in Section 8. Section 9 mentions briefly a slightly more general problem, that is selecting the events associated with the t highest probabilities without regard to order.

(R. Hooke)

Editorial Note: See also abstract No. 191.

BOX, G. E. P. & LUCAS, H. L. (Statist. Tech. Res. Group, Princeton and Inst. Statist., N. Carolina) 9.3 (4.0)
Design of experiments in non-linear situations—*In English*
Biometrika (1959) **46**, 77-90 (7 refs., 1 table, 4 figs.)

Suppose a response is a known (non-linear) function of k design variables (whose levels may be chosen in advance) and p unknown parameters. The problem is to choose N sets of values of the design variables so that from observations at these points the parameters may be estimated with high accuracy. This situation occurs for example in studying a consecutive chemical reaction; the yield of the intermediate product after any chosen time has elapsed involves the exponentials of multiples of the (unknown) rate constants.

In practice, the possible range of the design variables will be restricted; it is here assumed that experimentation can be carried out throughout a region bounded by inequalities in each design variable separately.

It is proposed to choose the design so that a certain determinant (Wilks' generalised variance) is minimised. The significance of this criterion is discussed. Preliminary estimates of the parameters are essential; the design is then to be chosen so that it would be optimal if these estimates were at the true values. Several extensions of the problem are mentioned.

Now N is taken to be equal to p. (If, in addition to estimating the unknown parameters, the experiment is to be used to check the adequacy of the model it should be possible to specify a wider model involving extra parameters to be estimated.) In a geometrical formulation, the optimal design is obtained by choosing N points on a certain " design locus " (determined by the model) which with the origin form the simplex of greatest possible volume.

Several examples are treated in detail; the numerical problems involved are discussed, and an iterative method is proposed for finding the optimum design.

For some simple models the design locus is a nearly straight line, so that *no* design is very efficient in estimating the parameters separately. Designs for determining the response itself throughout a region are not discussed.

(C. L. Mallows)

BROSS, I. D. J. (Cornell University Medical College) 9.9 (7.2)
Note on an application of the Schumann-Bradley table—*In English*
Ann. Math. Statist. (1959) **30**, 581-583 (3 refs.)

Previous results of Schumann and Bradley [*Ann. Math. Statist.* (1957) **28**, 902-920] are extended to the variance component model. A test criterion is developed to compare the sensitivity of experiments based on the ratio of variance ratios.

(R. L. Anderson)

CONNOR, W. S. & ZELEN, M. (Statist. Eng. Lab., National Bureau of Standards, Washington) 9.2 (9.1)
Fractional factorial experiment designs for factors at three levels—*In English*
Appl. Math. Series No. 54, *Nat. Bur. Stand.* (1959) 1-42 (9 refs., 4 tables)

This publication contains a collection of fractional factorial experiment designs for the case when all factors are at three levels. It is a sequel to No. 48 in the National Bureau of Standards *Applied Mathematics Series* which contains plans for factors at two levels. This new document catalogues forty-one fractional factorial experiment plans for experiments having four to ten factors and for fractions ranging from 1/3 to 1/243 of the possible treatment combinations for experimentation. The treatment combinations are arranged into blocks.

Experiment designs are given for:

The 1/3 fractional replicate for 4 factors in 3 and 9 blocks; for 5, 6 and 7 factors each in 3-, 9- and 27-block arrangements.
The 1/9 fractional replicate for 6, 7 and 8 factors each in 3-, 9- and 27-block arrangements.
The 1/27 fractional replicate for 7, 8 and 9 factors each in 3-, 9- and 27-block arrangements.
The 1/81 fractional replicate for 8 and 9 factors each in 3-, 9- and 27-block arrangements.
The 1/243 fractional replicate for 9 and 10 factors each in 3-, 9- and 27-block arrangements.

These experiment designs have been constructed so as (1) to have *no* main effects " aliased " (or confounded) with two factor interactions and (2) to have as few as possible two factor interactions " aliased " with other two-factor interactions.

Each experiment design contains:

(i) the treatment combination written out in full,
(ii) the fundamental identity, and
(iii) information as to those two-factor interactions which are " aliased " with higher-order interactions.

A section on analysis discusses the estimation and biases of the various main effects and interactions. An example is presented of a 1/9 fractional replicate of a six-factor experiment for a completely randomised and a 27-block design.

(M. Zelen)

DAVID, H. A. (Virginia Polytechnic Inst., & Univ. of Melbourne) 9.5 (9.6)
Tournaments and paired comparisons—*In English*
Biometrika (1959) **46**, 139-149 (10 refs., 3 tables)

This paper considers the situation where t elements are compared in pairs, using the language of pairs of players who either win or lose games played between them. Its purpose is to discuss and contrast the properties of a complete set of paired comparisons (a " Round Robin " tournament) with the " Knockout " tournament. In the latter case t is supposed reduced to 2^p by a set of play-offs. The players are then " drawn " randomly to play a p-round knockout tournament—so that there is a probability $2^r/(t-1)$ that a given pair of players are drawn to play together in the $(r+1)$th round if they each win in all preceding rounds. The general Bradley-Terry model is used so that there is a set of probabilities $\{\pi_{ij}\}$ where π_{ij} is the probability that the ith player beats the jth in a given game. A straightforward method of obtaining the probabilities $\{P_i^{(r)}\}$ that the ith player reaches the rth round is given, and these are given explicitly for $t = 8$ $(p = 3)$. An example is taken of the probit type where the eight players have " strengths " varying normally about the normal mean sample octiles, and the numerical values of the corresponding π_{ij} and $P_i^{(r)}$ are tabled.

The " Round Robin " tournament is considered in the most general case where the ith and jth players play n_{ij} games (independently), with different probabilities for different pairs of players, and the joint generating function is obtained for the total numbers of games won by each player (being the product of the corresponding binomial generating functions). In the simple case $n_{ij} = 1$, $\pi_{ij} = \frac{1}{2}$, the probabilities of given sets of scores are derived from the generating function and are tabulated for $t = 3(1)8$. The generating function is also given for the simple case when scores for each game may have one of the $r+1$ values j/r, $j = 0, \ldots, r$. The joint probability distribution function of sets of s players $(s<t)$ is written down. On this basis are discussed tests for significantly large values of (i) a given player's score, (ii) the difference between two such scores and (iii) the top score. In the latter case a general expression is derived by the Halley-De Moivre theorem for the cumulative probability distribution function of the top score and expressions are given for top scores differing from the maximum $(t-1)$ by 0, 1 and 2. These are tabulated for $t = 3(1)20$ and are seen to cover the upper 5 per cent. point for t up to 17.

It is suggested that knockout tournaments may be superior if it is required to pick out the top ranking element rather than a set of elements. No experimental data are considered in the paper.

(D. E. Barton)

FOLKS, J. L. (Iowa State College) 9.1 (7.3)

Analysis of quadruple rectangular lattice designs—*In English*
Biometrics (1959) **15**, 74-86 (4 refs., 7 tables)

A method of constructing four replicates of a rectangular lattice using four different confoundings is given. Using Kempthorne's notation an outline of the derivation of the analysis of variance is provided. Explicit formulae are given for all the required sums of squares and adjusted treatment effects. The derivations are based on the use of both interblock and intrablock variances. The author shows how to estimate these variances from the analysis of variance table.

Variances of the differences between two adjusted treatment means are listed for all nine possible combinations. An example is worked in great detail.

(C. Y. Kramer)

GRIDGEMAN, N. T. (Division of Applied Biology, Nat. Res. Council, Ottawa) 9.5 (1.3)

Sensory item sorting—*In English*
Biometrics (1959) **15**, 298-306 (17 refs.)

Consider a group of M items of which N possess a certain stable attribute that is objectively determinate but subjectively equivocal. The problem considered is that of sorting off by sensory perception the subgroup of N items possessing the attribute in question. A probabilistic model is described for $R(0 \leqq R \leqq N)$, the number of correct allocations to the N-sized subgroup. A distinction is drawn between the perception of the attribute in an item, and the correct allocation of the item. The sorted-off subgroup is considered to consist of some items identified as a result of perception, some items correctly identified as a result of chance selection, the remaining $N-R$ items being incorrectly allocated.

The model is described in terms of combinatorial probability, and an expression for the rth moment of the sorting distribution is obtained. It is observed that the model covers many sensory test designs including the simplest, viz. pair comparison ($M = 2$). In addition to the primary probability of sensory discrimination, certain secondary or component probabilities are considered. These are concerned with quantitative characterisation, intensity ranking, and preference as related to the sensations evoked by the stimuli. Brief reference is made to polytomous sorting (involving more than two subgroups).

(W. A. Glenn)

KIEFER, J. & WOLFOWITZ, J. (Cornell University, Itraca, New York) 9.4 (6.1)
Optimum designs in regression problems—*In English*
Ann. Math. Statist. (1959) **30**, 271-294 (15 refs.)

This is a general attack on the optimisation of design (i.e. determining " best " locations at which to take observations) in regression problems. Known results are collected, summarised and extended to form a rather general theory. Proofs are simplified by the use of known results from the theory of games, and results are made more useful by the provision of some devices which reduce the determination of optimum strategies to feasible computations.

The paper is divided into three parts. The first part (sections 2 and 3) treats the problem of estimating one regression coefficient, the others being unknown. The second part (section 4) treats the case of inference in connection with several coefficients, and the third part (section 5) the case of estimation of the whole regression function.

The regression situation is defined as follows: given any point x (value of the independent variable) in a compact space, X, one can observe a random variable Y_x which has the properties

$$E(Y_x) = \sum_{1}^{k} a_i f_i(x) \text{ and Var } (Y_x) = \sigma^2,$$

where the f_i are continuous, linearly independent, known functions; the a_i are unknown coefficients of the regression function; and σ^2 is constant but unknown. An integer n is given and a collection $X = (x_1, x_2, \ldots, x_n)$ of n not necessarily distinct points in X is selected at which the independent random variables $Y_{x_1}, \ldots, Y_{x_n}$ are observed. The collection X is equivalent to a measure ξ which assigns to each point X in χ a mass equal to $1/n$ times the number of appearances of x in $x_1, x_2, \ldots, x_n$. The optimisation problem is solved in terms of general discrete probability measures rather than just those which assign integral multiples of $1/n$ to points in X; once found, an optimum measure ξ^* must be approximated by a measure which assigns mass in integral multiples of $1/n$.

The problem of estimating one parameter, say a_k, proceeds as follows. First an expression is derived which is σ^2 times the inverse of the variance of the best linear estimator of a_k. From this follows the definition of an optimum design ξ^* as one which maximises the expression.

The determination of optimum designs is based on the concept of Tchebysheff approximation. Theorem 1 follows, providing a sufficient condition for an optimum design, and theorem 2 shows that the condition is also necessary and establishes the existence of an optimum strategy: theorem 3 gives some conditions for uniqueness.

Section 3 provides examples of the one-coefficient case discussed in section 2. Particular attention is given to the common case of polynomial regression in one variable. Here X becomes a closed, finite, nondegenerate interval $[\alpha, \beta]$ of real numbers, and the f_j are given by $f_j(x) = x^{j-1}$, $1 \leqq j \leqq k$, where $k = h+1 > 1$. The problem is considered to be that of assuming that the regression function is a polynomial of degree $\leqq h$ and testing the hypothesis that it is actually of degree $\leqq h-1$. The interval $[\alpha, \beta]$ is changed by a scale-and-location transformation to the interval $[-1, 1]$, in terms

KIEFER, J. & WOLFOWITZ, J. (Cornell University, Itraca, New York) 9.4 (6.1) *continued*
Optimum designs in regression problems—*In English*
Ann. Math. Statist. (1959) **30**, 271-294 (15 refs.)

of which the optimum strategy ξ^* is given by

$$\xi^*(-1) = \xi^*(1) = 1/2h$$
$$\xi^*(\cos j\pi/h) = 1/h \quad 1 \leqq j \leqq h-1.$$

It is pointed out that commonly used procedures taking equal numbers of observations at equally spaced points on the interval take at least 38 per cent. more observations to yield the same variance for the best linear estimator of the unknown parameter.

Section 4 takes up the case in which inference is to be made about more than one of the regression coefficients. Several criteria for optimality are available, the one selected being called " D-optimality ". A design is D-optimum if the specified coefficients are estimable and if the design minimises the determinant of the covariance matrix of the best linear estimators of these coefficients. Results concerning optimum designs are summarised in theorem 4.

Finally, section 5 is concerned with estimation of the whole regression function. Again, several criteria of optimality are available. The one selected defines an optimum design as one which minimises

$$\max_x E[\Sigma_i(\hat{a}_i - a_i)f_i(x)]^2.$$

A sufficient condition is found (its necessity to be proved in a later paper) and theorem 5 states that if this condition holds for a design concentrated on k points, then the design gives measure $1/k$ to each of the points.

(R. Hooke)

KSHIRSAGAR, A. M. (University of Bombay) 9.1 (7.3)

Distribution of the " Blocks adjusted for treatments " sum of squares in incomplete block designs—*In English*

Ann. Math. Statist. (1959) **30**, 246-249 (4 refs.)

This note fills a gap in the knowledge of distributions of quantities associated with incomplete block designs. The method is primarily one of matrix algebra and the two principal cases, where the block effects are fixed and random variates respectively, are studied for connected design, i.e. where the treatment block contrasts are capable of being estimated.

For a design, the dual of which is a balanced incomplete block, all the non-zero latent roots of the matrix (the difference between the diagonal matrices) are equal and the distribution in question reduces to a chi-square distribution. When the design is not a connected one, some of the roots will be zero and appropriate changes are easily made.

(R. L. Anderson)

MITTON, R. G. & MORGAN, F. R. (Brit. Leather Manufacturers' Res. Ass., Egham, England) 9.2 (11.1)

The design of factorial experiments : a survey of some schemes requiring not more than 256 treatment combinations—*In English*

Biometrika (1959) **46**, 251-259 (9 refs., 10 tables)

Four- or eight-level factors can be regarded as formally equivalent to two or three factors at two levels, and when only such factors are involved factorial designs are readily prepared. Designs become harder to find when there are many factors with two, four, or eight levels, and a number of such designs are given here, none of which requires more than 256 treatment combinations. Use is made of fractional replication (down to the sixty-fourth replicate) but not partial confounding, and it is assumed that second- and higher-order interactions can be used for the estimate of error and for confounding. However, main effects are unconfounded, and, where possible, first-order interactions also. In schemes with fractional replication, no main effect or first-order interaction has as alias an interaction of lower order than the second.

The tables are as complete as possible; a few of the schemes are similar to those given by other workers. The tables show:

(i) a coded reference number,
(ii) the generators of the defining contrasts,
(iii) the generators of the interactions confounded with blocks,
(iv) the generators of the principal block,
(v) multipliers for the other blocks,
(vi) the degrees of freedom for error,
(vii) the first-order interactions confounded with blocks.

(D. Harris)

PATTERSON, H. D. & LUCAS, H. L. (Rothamsted Exp. Sta. and Inst. Statist., North Carolina State College) **9.7** (7.3)
Extra-period change-over designs—*In English*
Biometrics (1959) **15**, 116-132 (11 refs., 4 figs., 8 tables)

Designs involving cyclical arrangements of treatments are frequently used in experiments extending over several periods in order to reduce errors due to variation between experimental units. These designs are known as change-over designs. Change-over designs which provide for the estimation of first residual effects (carry-over effects of previous treatment) have been devised using special Latin squares. Designs which permit the number of periods to be less than the number of treatments have also been investigated. All of these designs have the following faults:

(i) residual effects are less accurately determined than direct effects;

(ii) the estimated variance of the sums of direct and residual effects for individual treatments (i.e. permanent effects) are greater than the sums of the variances of the separate effects;

(iii) tests of direct effects and residual effects are not independent.

Extra-period designs in which the treatments of the last period of the basic Latin square design are repeated were developed to meet these criticisms. In these designs direct and residual effects are orthogonal.

The authors give the preceding as background and then define "basic change-over design" as any change-over design in which no treatment is applied to any experimental unit is more than one period, and which satisfy certain conditions of balance as given in the paper. These basic designs consist of t treatments in b blocks of k units, with p periods. Direct and residual effects are not orthogonal in the basic designs. This paper is primarily concerned with the extension of the basic designs to extra-period designs by repeating the treatments of the last period in an extra period.

In the special case $t = k = p$, it has been shown that for this type of extra-period design the direct and residual effects are orthogonal. The authors generalise this result to the basic designs with an extra-period (repeating the treatments in the last period) where t, k, and p are not necessarily equal. For this general class of extra-period designs it is shown that direct and residual effects are orthogonal. Four designs are illustrated and the analysis of the extra-period designs are described. As with the basic designs the estimates of error given by the method of fitting constants for direct and residual effects are biased. Nearly unbiased estimates of errors of residual effects can be obtained with certain types of designs. The efficiencies and uses of extra-period designs are discussed.

In conclusion the authors state that the designs are likely to be most useful when increased accuracy is required in the estimates of residual and permanent effects, but that they can be used to advantage

(*a*) to extend for one further period an experiment which would otherwise be terminated, and

(*b*) to provide balanced designs in blocks of two units.

(H. O. Posten)

RAGHAVARAO, D. (University of Bombay) **9.8** (9.1)
Some optimum weighing designs—*In English*
Ann. Math. Statist. (1959) **30**, 295-303 (7 refs.)

Suppose we have N objects to be weighed in N weighings with a balance having no bias. Let the design matrix $X(N\times N) = (x_{ij})$ where $x_{ij} = 1(-1)$ if the jth object is placed in the left (right) pan in the ith weighing, and $x_{ij} = 0$ if the jth object is not weighed in the ith weighing. In this paper, the best weighing designs are obtained for the cases (i) N odd and (ii) $N \equiv 2 \pmod 4$ subject to the conditions that the variances of the estimated weights are equal and the estimated weights are equally correlated. The word "best" is used in the manner of Kishen where efficiency is measured by $1 \sum_{i=1}^{N} C_{ii}$ where $(c_{ij}) = (X'X)^{-1}$.

For the case where N is odd it is shown that the best weighing design X is such that $X'X = (N-1)I_N + E_{NN}$ where I_N is the identity matrix of order N and E_{NN} is an Nth order matrix with positive unit elements everywhere. The efficiency of this design is $(2N \pm 1)/2N$. It is shown that a necessary condition for the existence of an Nth order matrix with elements $+1$ and -1 satisfying the above equation is that $N = (d^2+1)/2$ where d is an odd integer. Also, if a balanced incomplete block design exists with parameters

$v^* = b^* = N$, $r^* = k^* = (N \pm d)/2$, $\lambda^* = (N \pm 2d + 1)/4$,

then by changing the zeros into -1's in the incidence matrix of the incomplete block design, such a weighing design as described above is obtained.

When $N \equiv 2 \pmod 4$ it is shown that the best weighing design X is that for which $X'X = (N-1)I_N$, the efficiency of which is $(N-1)/N$. For the case where $N = p^h + 1$ where p is an odd prime and h is a positive integer such that $p^h \equiv 1 \pmod 4$, then one can construct such design matrices based on Galois Fields and the Legendre function ξ, the method of construction being given in one of the listed references.

There are listed seven design matrices for the cases where $N = 5$, 6, 10, 13, 14, 18 and 25. For each of these matrices the author provides the variance of each estimated weight, the covariance of each pair of estimated weights, and the efficiency.

(J. W. Wilkinson)

ROBERTSON, A. (University of Edinburgh) 9.4 (7.2)

Experimental design in the evaluation of genetic parameters—*In English*
Biometrics (1959) **15**, 219-226 (3 refs., 1 table, 3 figs.)

Optimum population structure for the estimation of intraclass correlations and genetic parameters by analysis of variance methods is discussed. Starting with Fisher's formula for the sampling variance of the intraclass correlation, both the single classification and that involving combined estimates from sire and dam components are considered. Further, the sampling variances of heritability estimates derived from intraclass correlations are compared with those for parent-offspring estimates.

In general, the sampling variance is a joint function of sample size and the intraclass correlation coefficient itself. For the single classification, this variance is at (or very near) a minimum when the product of the group size and the intraclass correlation coefficient is unity. The optimum group size becomes the reciprocal of the intraclass correlation. When, for example, the intraclass correlation lies between 0.01 and 0.1 in numerical value, the optimum group size should be 20 to 30. Extremely small groups should be avoided, since the sampling variance rises rapidly as group size decreases below 10.

The use of both sire and dam components of variance (as from a hierarchical analysis) necessitates a compromise. For estimates obtained by use of only the sire component, minimum sampling variance of the intraclass correlation coefficient is achieved with one offspring per dam and the number of dams per sire equal to the reciprocal of the correlation itself. For estimates from use of the dam component, minimum variance results when the number of offspring per dam is the reciprocal of the correlation. When estimates from both sire and dam components are combined, the family size for each dam should be one-half the reciprocal of the intraclass correlations with 3 or 4 dams per sire. If the dam component includes maternal effects, its use may be inadvisable.

When the heritability ratio is 0.25, estimates from intraclass correlations among half-siblings should be equal in efficiency to parent-offspring estimates. For values lower than 0.25, the half-sib method should be more efficient.

As the author indicates, this paper deals with experimental situations having groups of equal size. He also gives a word of caution regarding estimates of variances of intraclass correlations when the correlations themselves are very small numerically. As a consequence, the formulas shown in the paper may not be appropriate for use on field data, particularly when subgroups are unequal in size. Further, in the cases where heritabilities are very low (e.g. less than 0.10), the formulas may not be applicable at all.

(N. R. Thompson)

ROY, J. (University of North Carolina, Chapel Hill) 9.1 (9.4)

A class of two replicate incomplete block designs—*In English*
Biometrics (1959) **15**, 259-269 (14 refs., 4 tables)

A new class of two replicate designs called simple partially linked block designs is introduced. The properties of these designs are:

(i) Each treatment occurs at most on one plot in a block and altogether on two plots,

(ii) Any two blocks have at most one treatment in common,

(iii) Two blocks are first (second) associates if they have one (no) treatment in common and this association scheme is partially balanced with parameters n_1, n_2, and $p^i_{jk}(i, j, k = 1, 2)$.
Thus, $v = \frac{1}{2}n_1(n_1+n_2+1)$, $b = (n_1+n_2+1)$, and $k = n_1$.

A list of simple partially linked block designs with $k \leqq 10$ is given as well as the values of the parameters required to analyse these designs by the P-method described by Rao. These designs are highly efficient with an efficiency-factor of the order of 75 per cent.

It is shown that, with any of these designs, the variance of the estimate of the difference in effects of two treatments can be at most of seven different types and these are given explicitly.

This class of designs may be constructed by dualising any partially balanced association scheme with two associate classes and parameters n_1, $n_2 p^i_{jk}(i, j, k = 1, 2)$ with $k^* = 2$ constructed by forming one block with each pair of treatments that are first associates.

A numerical example is given to illustrate the calculations involved to obtain the analysis of variance, variances of the differences in effects of two treatments, and the efficiency of the design. The example has $v = 15$, $b = 10$, $k = 3$, and, of course, $r = 2$.

(C. Y. Kramer)

SHAH, B. V. (University of Bombay) 9.1 (0.6)

On a generalisation of the Kronecker product designs—*In English*

Ann. Math. Statist. (1959) **30**, 48-54 (6 refs., 1 table)

In order to use the results of this paper it is necessary to have a knowledge of the notation and definitions available, for example, in works of Bose, Nair, and Rao as referenced in this paper. The authors denote the parameters of balanced incomplete block designs by v^*, b^*, r^*, k^*, λ^* and the parameters of a partially balanced incomplete block by $v, b, r, k, \lambda_1, \ldots, \lambda_m, n_1, \ldots, n_m, p_{ij}^k$.

Fisher and Yates provide a complete table of balanced incomplete block designs for $r, k \leqq 10$. This paper provides a method of constructing a special class of partially balanced incomplete block designs from the known balanced incomplete block designs. All partially balanced incomplete block designs in this paper are of three associate classes with the rectangular association scheme for $v = pq$ treatments written in p rows and q columns so that treatments in the same row are the first associates, treatments in the same column are the second associates and the remainder are third associates.

The balanced incomplete block designs included have parameters

1. $v^*, b^*, r^* = k^* = \lambda^* = 0$.
2. $v^* = k^*, b^* = r^* = \lambda^*$.
3. $v^* = b^*, r^* = k^* = 1, \lambda^* = 0$.

The construction of the partially balanced incomplete block designs is based on certain "balanced matrices" which are tabulated. The balanced incomplete block designs used as a basis for v, $b \leqq 15$ and $S \leqq 3$ (the elements of the matrix range over integers not greater than S) are certain balanced incomplete block designs with

v^*	b^*
3	3, 6, 9, 12, 15
4	12
5	10
6	15
7	7, 14
9	12
11	11
15	15

for which balanced matrices can be constructed. The author states that any other balanced matrix with v^*, $b^* \leqq 15$ and $S \geqq 3$ does not exist since corresponding balanced incomplete block designs do not exist.

The author also states that the method given differs from that of Vartak and Sillito only in the fact that two or more distinct elements of a matrix are replaced by different matrices as contrasted to the replacement of two distinct elements of a matrix by two distinct matrices.

(W. J. Dixon)

SHRIKHANDE, S. S. (University of North Carolina) 9.1 (–.–)

On a characterisation of the triangular association scheme—*In English*

Ann. Math. Statist. (1959) **30**, 39-47 (3 refs., 1 table)

Consider an incomplete block design for v treatments in blocks containing k (less than v) treatments, such that each treatment occurs the same number of times in the design and such that any two treatments occur together in either λ_1 or λ_2 blocks of the design, where λ_1 and λ_2 are different non-negative integers. It is shown that the "association scheme" of the design is of a particular type known as a triangular association scheme, as explained in the last two papers listed in the references, if

(i) $v = n(n-1)/2$ and $n = 5$ or 6;
(ii) each treatment occurs with exactly $2n-4$ treatments in λ_1 blocks and with the remaining treatments in λ_2 blocks;
(iii) with respect to any two treatments which occur together in λ_1 blocks there are exactly $n-2$ treatments every one of which occurs with each of these two treatments in λ_1 blocks;
(iv) with respect to any two treatments which occur together in λ_2 blocks, there are exactly 4 treatments everyone of which occurs with each of these two treatments in λ_1 blocks.

Connor [*Ann. Math. Statist.* (1958) **29**, 262-266] derived an equivalent result for $n \geqq 9$.

(S. S. Shrikhande)

SPROTT, D. A. (Waterloo College, Ontario) **9.1** (–.–)

A series of symmetrical group divisible incomplete block designs—*In English*
Ann. Math. Statist. (1959) **30**, 249-251 (5 refs.)

This note shows how a series of group-divisible designs—as defined in 1952 by Bose and Connor [*Ann. Math. Statist.* **23**, 367-383]—can be obtained from the series of balanced incomplete block designs given by Bose [*Ann. Eugenics* (1939) **10**, 352-399]. The condition for a balanced incomplete block design to be symmetrical is given as $v = b(r = k)$ where the design concerns v (elements) in b (blocks) of k different elements in each block, so that each element occurs in r blocks.

(R. L. Anderson)

STONE, M. (British Med. Res. Council Appl. Psych. Unit) **9.4** (6.1)

Application of a measure of information to the design and comparison of regression experiments—*In English*
Ann. Math. Statist. (1959) **30**, 55-70 (10 refs.)

A normal regression experiment can be represented by

$$Y_i = \sum_{j=1}^{k} X_{ij}\theta_j + \eta_i \qquad (i = 1, \ldots, n)$$

where $\{\eta_i / i = 1, \ldots, n\}$ is a set of normally distributed random variables with zero means and a non-singular dispersion matrix C; $\theta = (\theta_1, \ldots, \theta_k)$ is the parameter-vector of interest and $X = (X_{ij})$ is a known $n \times k$ matrix which will be called the allocation matrix. The rows of X will be called the allocation vectors. We denote the experiment by $\epsilon(X, C)$. We assume that C is known; generally it will be function of X, $C(X)$. The particular realisation of Y will be denoted by y. The matrix $F = X'C^{-1}X$ is the Fisher-information-matrix of $\epsilon(X, C)$.

When F is non-singular, one answer to the question "What information does y give about θ?" is to quote F^{-1}, the dispersion matrix of the maximum-likelihood-estimates of θ. A strong argument in favour of this is that F^{-1} is independent of both θ and y. The fact that it is independent of θ means that the answer is not "local"; the fact that it is independent of y leads to simplicity. This approach was taken in 1957 by Box and Hunter [*Ann. Math Statist.* **28**, 195] in their work on rotatable designs. However, many experimenters wish to have a one-dimensional answer to the question. In 1952, Elfving [*Ann. Math. Statist.* **23**, 255] developed the use of trace F^{-1}. This paper adopts the measure of information introduced in 1956 by Lindley [*Ann. Math. Statist.* **27**, 986] and generalises that treatment of the regression situation to include the singular case. It explains the uses of the measure and compares it with that of Elfving and provides links with the traditional variance approach. An asymptotic form of the measure for large n is obtained and related to Neyman-Pearson theory. The influence of nuisance parameters is discussed and an analog of a theorem of Chernoff is established.

The information measure is defined in the Bayesian framework: the gain of information is defined as the functional $\Delta I = Ip(\theta/y) - Ip(\theta)$ where $Ip(u) = \int p(u) \log p(u) du$.

The expression for ΔI proves useful in three ways:

1. If we decide to experiment until the gain of information reaches a certain level then the fact that ΔI is independent of y allows us to state in advance whether a particular experiment will give us the required gain. Among experiments which do, we may choose the one which is most economical in some sense. This is the case of fixed-sample-size experimentation.

2. Two experiments can be compared in the following sense:

"Any result of $\epsilon(X_1, C_1)$ will give $\Delta I_1 - \Delta I_2$ more information than any result of $\epsilon(X_2, C_2)$."

3. If we have a choice of performing any experiment from a given class, we may choose the $\epsilon(X, C)$ which maximises ΔI.

(W. J. Dixon)

ANDERSON, O. Jr. (*Ifo*-Institut für Wirtschaftsforschung, Munich) **10.0** (10.8)

The decomposition of time series from the point of view of applied business cycle research—*In German*

Allg. Statist. Arch. (1958) **42**, 358-367 (6 refs., 5 figs.)

The paper is an extract of a lecture delivered to the general assembly of the German Statistical Society.

The problem of decomposing a series of economic observations in components (trend, cyclical, seasonal and irregular component) can be tackled in two different ways:

(*a*) Each of the different components is linked with some specific system of causes by a mathematical model (e.g. the seasonal movement of building activity is expressed as a function of the weather-conditions). A useful technique for this purpose is multiple regression analysis.

(*b*) The different components are "extracted" by means of their characteristic development in time (trend+cyclical; smooth; seasonal; periodic).

The application of method (*a*) is illustrated by eliminating the seasonal component from a series depicting the production of electrical energy. The seasonal variation in the production of electrical energy is assumed to depend primarily on two factors: the length of the days and the general level of business activity. Therefore, it is expressed as a function of two variables: the month—determining the length of days—and the 12-months' moving average—reflecting the general level of business activity. It is shown that for every month, the production of electrical energy can be presented with sufficient accuracy as a linear function of the 12-months' moving average. In this way, the seasonal component can be eliminated and a series consisting of the smooth and an irregular component be obtained.

In order to judge whether the actual development is consistent with some assumptions concerning the development of the smooth component a confidence bond is constructed by means of the standard deviation obtained from the deviations of the actual values from their regression line. As long as the observations represented by the smooth and the irregular component (calculated from the observed data) moves within this belt, its development is consistent with the assumption concerning the development of the smooth component.

(J. Pfanzagl)

BLYTH, C. R. (Univ. of Illinois and Stanford Univ.) **10.5** (4.3)

Note on estimating information—*In English*

Ann. Math. Statist. (1959) **30**, 71-80 (4 refs., 3 tables)

Consider a random variable Y with k distinct possible values $a_1, \ldots, a_k$ and associated probabilities $p_1, \ldots, p_k$. An appropriate non-metric dispersion measure for Y which assesses the extent to which the total probability is broken up into pieces in the distribution of Y, irrespective of any possible distance function in the space of Y, is the entropy or information (in the Shannon-Weaver sense) defined by

$$H = -\sum_{1}^{k} p_i \log_2 p_i.$$

The function H is, in fact, the only continuous non-metric dispersion measure $f_k(p_1, \ldots, p_k)$ except for a constant multiplier), with $f_k(1/k, \ldots, 1/k)$ an increasing function of k, which has the following desirable addition property: if the values of Y are divided into groups, then the dispersion of Y is equal to the "between group" dispersion plus the expected "within-group" dispersion.

The problem considered is that of estimating H from n independent observations of Y. Let X_i be the number of observations with the value a_i. Then since $X_1, \ldots, X_k$ is a sufficient statistic for $p_1, \ldots, p_k$ with a multinomial family of possible distributions any reasonable estimator must be a function of $X_1, \ldots, X_k$ only. Again, since the expectation of any function of $X_1, \ldots, X_k$ is a polynomial in $p_1, \ldots, p_k$ of degree $\leqq n$ the only functions of $p_1, \ldots, p_k$ which admit of unbiased estimation are polynomials of degree $\leqq n$. Hence and in particular, no unbiased estimator exists for H. Further, since the family of multinomial distributions represents a complete class, the unique unbiased uniformly minimum variance estimator for any polynomial of degree $\leqq n$ in $p_1, \ldots p_k$ is immediately available.

The previous remarks suggest the approximation of H by a polynomial H_r in $p_1, \ldots, p_k$ of degree $r \leqq n$ from a suitable Taylor expansion and the estimation of H by the uniformly minimum variance unbiased estimator Z_r of H_r. Naturally, Z_r, considered as an estimator for H, will be more or less biased to the degree that H_r is an approximation of H. Upper and lower bounds for the bias, which for $k > 2$ is positive, are determined and an exact expression for the variance of Z_r is obtained. Actually for $k > 2$ a suitably improved estimator Z_r' rather than Z_r is used to reduce the bias. The corresponding results for the asymptotic case ($n \to \infty$) are also derived.

A comparison with the maximum likelihood estimator shows that Z_n is much less biased than the maximum likelihood estimator for large n (except when the entire probability is concentrated at a single value) and is also asymptotically efficient. For small n, the maximum likelihood estimator is poor from the point of view of mean squared error and is also more biased than Z_n over most of the range of $p_1, \ldots, p_k$. Comparison of the low-bias estimate Z_n' with the minimax estimate, for squared error loss function and $k = 2$, shows that Z_n' performs creditably even when n is as low as 2 or 3. Finally, a low-bias estimator is derived when the number of classes k is unknown.

(H. Ruben)

BOCHNER, S. & KAWATA, T. (Princeton Univ.) **10.6** (1.6)

A limit theorem for the periodogram—*In English*

Ann. Math. Statist. (1958) **29,** 1198-1208 (4 refs.)

This is a study of the asymptotic properties of a generalised (i.e. weighted or smoothed) periodogram of a real stationary (wide sense) stochastic process $\epsilon(t)$ with absolutely continuous spectral distribution function. It is assumed that $\epsilon(t)$ is stationary even of fourth order, so that the expectation

$$\mathscr{E}[\epsilon(t)\epsilon(t+s_1)\epsilon(t+s_2)\epsilon(t+s_3)] = P(s_1, s_2, s_3)$$

exists, and is a function only of s_1, s_2, s_3. Further, P is the sum of two functions: one, $P_G(s_1, s_2, s_3)$, having the general form of the fourth moment of a Gaussian process; the other, $Q(s_1, s_2, s_3)$ being Lebesque integrable in Euclidean 3-space, with Fourier transform having the same property. The function Q is identically zero if the process $\epsilon(t)$ is stationary Gaussian. Then, under certain regularity conditions on the periodogram weighting function and its Fourier transform, it is shown that

(1) the covariance of weighted periodograms computed over two time-intervals becomes in the limit, as the time-intervals tend to infinity in fixed (non-zero and non-infinite) ratio, equal to a constant multiple of the square of the process spectral density (different constants applying for zero and non-zero values of the argument of the density function),

(2) the expectation of the square of the difference of the weighted periodograms becomes, in the same limit, a constant multiple of the square of the spectral density [different constants for zero and non-zero values of the argument of the density function; constants different from those in (1)].

(D. P. Gaver, Jnr.)

CONOLLY, B. W. **10.4** (–.–)

The busy period in relation to the queueing process **GI/M/1**—*In English*

Biometrika (1959) **46,** 246-251 (5 refs., 1 table)

The queueing process considered has one server dealing, in order of arrival, with a queue which has a general independent input. The service has a negative exponential probability density function; inter-arrivals and service intervals are statistically independent.

In this paper, the length of a busy period (when the server is fully occupied) is measured simultaneously in time and in the number served. In order to make probability statements about these lengths, they are investigated using an integro-difference equation technique. A generating function for the Laplace transforms of the required probability density functions is found. Limiting values of this function, and hence the average number of customers served in a busy period of finite duration, are then obtained for all values of the congestion index. The effect on the length of a busy period of regularising the input is also considered, and numerical comparisons between the two processes are given in a few typical cases. It appears that regularising the arrivals increases the probabilities of shorter busy periods.

(D. Harris)

FERSCHL, F. (Bundeskammer der gewerblichen Wirtschaft, Vienna) **10.4** (2.5)
Queueing models—*In German*
Math., Tech., Wirtschaft (1959) **3**, 98-102 (5 refs., 2 tables, 7 figs.)

This article gives a first introduction into the main problems and the most important viewpoints of queueing theory. It contains a summary of formulas for the case of service channels in parallel with Poisson arrival distribution and exponential service time distribution. An enumeration of a number of widespread fields of applications concludes the report. Seven very instructive figures facilitate the understanding of the concise and clearly written text.

(W. Eberl)

GANI, J. & PRABHU, N. U. (Univ. of Western Australia and Karnatak Univ.) **10.4** (1.1)
Remarks on the dam with Poisson type inputs—*In English*
Aust. J. Appl. Sci. (1959) **10**, 113-122 (14 refs.)

This paper reviews some aspects of the continuous-time model for a dam of finite capacity $K<\infty$, fed by simple Poisson inputs of fixed magnitude $h>0$ and with a constant parameter $\lambda>0$. There is a steady release except when the dam is empty, and an overflow occurs whenever the content exceeds K. It is pointed out that two apparently distinct results of Moran [*Aust. J. Appl. Sci.* (1955) **6**, 117-130] and Gani [*Biometrika* (1955) **42**, 179-200] obtained by the application of limiting methods to discrete dam models are in effect identical, and provide the stationary solution to this process.

For an infinite dam ($K\to\infty$), the forward Kolmogoroff equation is known to be a special case of Takács's [*Acta Math. Acad. Sci. Hung.* (1955) **6**, 101-129] integro-differential equation. It is shown that for the finite dam ($K<\infty$), the Kolmogoroff equation retains this identical form in the range $(0, K)$ of the dam content.

The known stationary distribution function for the content of the infinite dam is obtained directly by solving a difference-differential equation which is the limiting form (as time $t\to\infty$) of the Kolmogoroff equation. A step-by-step solution to the equation is readily found; this stationary solution exists only for $\lambda h<1$.

For the finite dam, an identical difference-differential equation for the stationary distribution function of the content holds in the range $(0, K)$. Its solution is none other than that for the infinite dam, truncated at K and appropriately scaled; this stationary solution is valid for all $\lambda h>0$.

The paper concludes with a brief sketch of new results for the dam with continuous inputs having certain infinitely divisible distributions conveniently described as of "Poisson" type. The forward Kolmogoroff equation for such processes is given, and its formal solution stated. The stationary case is then considered, and using the Kolmogoroff equation, it is shown that Moran's results [*Quart. J. Math.* (1956) **7**, 130-137] for the dam with inputs having a Gamma distribution can be readily obtained.

(J. Gani)

GOODMAN, L. A. (University of Chicago) **10.3** (5.2)

On some statistical tests for *m*th order Markoff chains—*In English*

Ann. Math. Statist. (1959) **30**, 154-164 (19 refs.)

This paper considers various statistics for testing hypotheses associated with positively regular Markoff chains, and certain asymptotic relations holding between them.

If $\{X_1, X_2, \ldots, X_n\}$ is an observed realisation from a stochastic process which, for convenience, is considered to be circularised so that X_1 follows X_n, then the frequency of a particular sequence $\underline{u}_s = (u_1, u_2, \ldots, u_s)$ of length s (an s-tuple) in the circularised realisation is written $f(\underline{u}_s)$. The statistic $\psi^2_{1,s}$ to test the hypothesis $H(P_1)$ that the process is a positively regular first-order Markoff chain with a states, having a constant transition probability matrix $P_1 = (p_{ij})$ is given by

$$\psi^2_{1,s} = \sum_{\underline{u}_s} [f(\underline{u}_s) - f_1(\underline{u}_s)]^2 / f(\underline{u}_s),$$

where $f_1(\underline{u}_s) = np'_{u_1} \Pi_{i=1}^{s-1} p_{u_i u_{i+1}}$ is the asymptotic expected value of $f(\underline{u}_s)$ given $H(P_1)$, and p'_{u_1} denotes a stationary probability. The number of possible s-tuples which can be formed given P_1 is denoted by k_s. It is known that for $s \geqq 2$, the statistics $\Delta\psi^2_{1,s} = \psi^2_{1,s} - \psi^2_{1,s-1}$ are distributed asymptotically ($n \to \infty$) as χ^2 with $\Delta k_s = k_s - k_{s-1}$ degrees of freedom, while for $s \geqq 3$, $\Delta^2\psi^2_{1,s} = \psi^2_{1,s} - 2\psi^2_{1,s-1} + \psi^2_{1,s-2}$ are asymptotically independent and distributed as χ^2 with $\Delta^2 k_s = k_s - 2k_{s-1} + k_{s-2}$ d.f.

A second statistic

$$\phi^2_{1,s} = 2\sum_{\underline{u}_s} f(\underline{u}_s) \log [f(\underline{u}_s)/f_1(\underline{u}_s)]$$

is defined, together with related statistics

$$L_{1,s} = \Delta\phi^2_{1,s} = \phi^2_{1,s} - \phi^2_{1,s-1} (s \geqq 2)$$

and $M_{s-2,s} = \Delta^2\phi^2_{1,s} = \Delta L_{1,s}$. Further statistics

$$K_s = 2\sum_{\underline{u}_s} f(\underline{u}_s) \log f(\underline{u}_s), \quad K_{1,2} = 2\sum_{\underline{u}_2} f(u_2) \log p_{u_1 u_2},$$

$K_{1,1} = 2\sum_{u_1} f(u_1) \log np'_{u_1}$ are also given, such that

$$\phi^2_{1,s} = K_s - (s-1)K_{1,2} - K_{1,1}, \quad \Delta\phi^2_{1,s} = \Delta K_s - K_{1,2},$$

and $\Delta^2\phi_{1,s} = \Delta^2 K_s$. Given $H(P_1)$, the $\phi^2_{1,s}$ are seen to be asymptotically equivalent to the $\psi^2_{1,s}$, while the

$$\Delta\phi^2_{1,s} = \Delta K_s - K_{1,2} = L_{1,s} (s \geqq 2)$$

are asymptotically χ^2 with Δk_s d.f., and

$$\Delta^2\phi_{1,s} = \Delta L_{1,s} = \Delta^2 K_s = M_{s-2,s}$$

are asymptotically independent and distributed as χ^2 with $\Delta^2 k_s$ d.f.

The statistics $L_{1,s}$ are the likelihood ratio statistics for testing the null hypothesis $H(P_1)$ within the alternate hypothesis H_{s-1} that the positively regular Markoff chain is of order $s-1$, while the $M_{s-2,s}$ are likelihood ratio statistics for testing the null hypothesis H_{s-2} within H_{s-1}. Under certain conditions, the asymptotic equivalence of $L_{1,s}$, $\Delta\psi^2_{1,s}$, and a further associated statistic $G_{1,s}$ is discussed, as well as the asymptotic equivalence of $M_{s-2,s}$, $\Delta^2\psi_{1,s}$ and $\Delta G_{1,s}$.

The paper concludes with a discussion of generalisations $\psi^2_{t,s}$, $\phi^2_{t,s}$ of the statistics $\psi^2_{1,s}$, $\phi^2_{1,s}$ as well as their related statistics. (J. Gani)

KARLIN, S., MILLER, R. G. Jr. and PRABHU, N. U. (Stanford Univ. and Karnatak Univ.) **10.4** (2.5)

Note on a moving single server problem—*In English*

Ann. Math. Statist. (1959) **30**, 243-246 (2 refs.)

The work reported in this note was done independently by the first two authors together and by the third author.

In 1959, McMillan and Riordan [*Ann. Math. Statist.* **28**, 471-478] stated the problem of a single moving server in terms of an assembly line moving at uniform speed with one attendant who moves with the line while working and against it with infinite velocity while transferring to the next item. The line has a barrier into which the server can be absorbed —and the service disabled—if service on a particular item is not completed within a given time period, i.e. distance of travel of the assembly line. McMillan and Riordan derived the generating function for the probability distribution of the items completed before the single server is absorbed in two special cases. This note validated and generalises that work by further exploiting the analogy to a simple queue with a single server. The time until absorption (in the moving single server problem) is equivalent to the length of a busy period (for the simple queue with a single server) where service distribution for the first item in line is $F_T(t) = 1$, $t > T$ and $F_T(t) = 0$, $t \leqq T$.

(R. L. Anderson)

KELLERHALS, G. (Math. Inst. Techn. Hoschsch, Stuttgart) 10.1 (10.0)

Boundary problems of conditional probabilities—*In German*

Math. Zeit. (1959) **71**, 1-30 (5 refs.)

Following Bartlett [*An Introduction to Stochastic Processes*, Cambridge, 1955], a discrete stochastic process which takes only values 0 and 1 and is defined at times 0, 1, ..., n is called a discrete alternative nearest neighbour process (system) if the conditional probability of an event A taking place at time x_ν (given events at times x_0, ..., $x_{\nu-1}$, $x_{\nu+1}$, ..., x_m) is equal to the conditional probability of A (given the events at times $x_{\nu-1}$ and $x_{\nu+1}$ only (linkage probability)) for all $x_0<\ldots<x_\nu<\ldots<x_m$. The process is said to be " open " if the probability of an event at time 0 given events at times n and x with $1\leqq x\leqq n-1$ does actually not depend on the event at time n. The process is " closed " if this is not true for any cyclic permutation of the time parameters. The class of all open discrete alternative nearest neighbour processes is essentially the class of all alternative Markoff chains although, as the author points out, in the case of Markoff chains there is a preferred direction of time contrary to the concept of a nearest neighbour process.

The main subject of the first part of the paper is the construction of a nearest neighbour process from its linkage probabilities, and, in the case of an open process, its " boundary probabilities ". They determine the process uniquely and under certain compatability conditions they can be arbitrarily prescribed. The compatibility conditions are stated in terms of probability ratios, i.e. the ratio of the probabilities of an event and its negation, and the proof is carried out with the help of recursion formulae for these ratios. The case of homogeneous open nearest neighbour processes, i.e. with time independent linkage probabilities, is investigated in detail. Every homogeneous Markoff chain represents such a process. Conditions for the converse statement are given in the sense that it is always true asymptotically for large n if the boundary probabilities are suitably altered.

In the second part of his paper the author considers continuous alternative nearest neighbour processes, defined in the same way, with time x ranging in an interval $0\leqq x\leqq d$. The recursion formulae of the discrete case are replaced by functional equations which are analogous to the Chapman-Kolmogoroff equations.

Assuming certain continuity and differentiability hypotheses on the linkage probabilities, differential equations are obtained which correspond to the backward and forward equations for Markoff processes. Their solutions furnish the linkage probabilities from which the process itself and transition probabilities are finally derived.

(K. Krickeberg)

LOMNICKI, Z. A. & ZAREMBA, S. K. (Boulton Paul Aircraft, Ltd., Wolverhampton) 10.1 (10.6)

On some moments and distributions occurring in the theory of linear stochastic processes II—*In English*

Monatsh. Math. (1959) **63**, 128-168 (12 refs., 1 table)

In the first part of their paper [*Monatsh. Math.* (1957) **61**, 318-358] the authors considered a linear stochastic process $x_t=\Sigma_k h_{t-k}\epsilon_k$ where k, $t=0,\ \pm1,\ \pm2,\ \ldots$ and $h_i=0$ whenever $i<0$. Depending on the theorem to be proved, it was assumed that $\{\epsilon_t\}$ is a wide (or strict) sense stationary process with zero mean, constant variance and successively uncorrelated or independent random variables; and $\Sigma|h_i|$ converges. With the help of a formula for the mixed moments of the x_t's, various asymptotic expressions for the moments of the arithmetic mean, and the univariate or mixed moments and the distribution of the covariance estimators were derived. In particular, if $\{\epsilon_t\}$ is stationary in the strict sense—the ϵ_t's being independent with finite fourth moments—and if we put

$$R_k=E(x_tx_{t+k}),\ R_{k,\,N}=N^{-1}\sum_1^N x_tx_{t+k},\ C_{k,\,N}=R_{|k|,\,N-|k|}$$

then, for any fixed set of integers $k(1)$, ..., $k(1)$, the distribution of each of the two sets of random variables

$$N^{\frac12}(C_{k(\nu),\,N}-R_{k(\nu)}) \text{ and } N^{\frac12}(R_{k(\nu),\,N}-R_{k(\nu)}),\ \nu=1,\ \ldots,\ 1$$

converges (when $N\to+\infty$) to a normal distribution. The covariance matrix of this normal distribution is given in terms of the R_k's and the second and fourth cumulant of the ϵ_t's.

The authors remarked that this still applies if an unknown polynomial trend is first eliminated and the covariances estimated from the residuals. Similar asymptotic results, under the same assumptions, hold for the estimators, of the autocorrelation functions.

These results are applied to statistical spectral analysis, and this is continued in the present, second part of the paper where the method allows to assume throughout only the existence of those moments which are actually required in the relevant statistics. First, asymptotic formulae —amounting to asymptotic normality—are given for the moments and distributions of the truncated or " raw " periodograms and related expressions, under a condition which is equivalent to consistency and which is shown to be indispensable. Similar statements apply to the reduced versions of the spectral density and distribution function obtained by substituting the autocorrelation function for the covariance function. Next the quadratic mean (over the frequency range $\frac12\leqq\omega\leqq\frac12$) of the difference between the spectral density function and any of these estimators is discussed in detail. It is shown that these " norms " are also asymptotically normally distributed.

Finally, various estimators for $\Sigma_k R_k^2$ are investigated. The precise asymptotic distribution of an estimator proposed by Grenander and Rosenblatt is obtained. However, the author's discussion of this estimator leads to the conclusion that it is not altogether useful. Therefore, they propose a different class of estimators whose asymptotic distributions are evaluated and found to be normal.

(K. Krickeberg)

MERCER, A. & SMITH, C. S. (Atomic Weapons Res. Estab., Aldermaston, and National Coal Board, London) **10.1** (1.5)
A random walk in which the steps occur randomly in time—*In English*
Biometrika (1959) **46**, 30-35 (2 refs., 1 table)

Suppose the steps of a one-dimensional random walk occur randomly in time at a fixed mean rate. The sizes of the steps are independent positive random variables with identical distribution. The distribution of the time taken to reach a given fixed barrier is investigated. The Laplace transforms of the moments of this distribution are obtained, and a non-rigorous argument leads to their asymptotic forms: these are given explicitly up to the third. It is stated that a rigorous proof can be obtained using results of W. L. Smith [*Biometrika* (1959) **46**, 1-29].

The problem arose from the study of conveyor belting, where wear may be caused both by continuous abrasion and by relatively infrequent blows. The model assumes that damage accumulates up to a certain level, when the belt is considered to be completely worn out.

Three examples illustrate the theoretical results. First, if the size of step has a negative exponential distribution, the first three moments of the time taken to reach the barrier are given explicitly; the asymptotic results are exact. If each step is the sum of two independent exponential variables, or if all steps are of equal size, the asymptotic results are not exact.

Now suppose the barrier to be moving at a fixed rate towards the origin; this corresponds to the continuous wear of the conveyor belting. A non-rigorous argument suggests the asymptotic forms of the first three " lifetime " moments; these depend on the moments of the step-size distribution, the mean frequency of steps, the initial distance of the barrier from the origin and a parameter measuring the relative importance of the discrete blows and the continuous wear. This particular parameter is of interest in the application considered. Assuming constant size of step, a numerical comparison is made between the asymptotic and exact values of the first three lifetime moments; it is shown that in order to estimate the above parameter it is necessary to solve a certain quartic equation.

(C. L. Mallows)

Editorial Note: see also abstract No. 315.

MOGYÓRODI, J. (Etövös University, Budapest) **10.1** (2.5)
Probabilistic treatment of the motion of neutrons in nuclear reactors—*In Hungarian*
Magy. Tud. Akad. Mat. Kut. Intézet. Közleményei (1959) **3**, 237-250 (4 refs.)

The author considers the spatial problem of the slowing-down to the termic level of neutrons in a reactor. He examines the path-length of the neutron in the slowing-down medium till the neutron becomes slowed down. The treatment is based on the following model: let us suppose the slowing-down medium, consisting of a homogeneous mixture of nuclei of k different types, to be infinite. Let the initial energy of the neutron be E_0, its actual energy E, its lethargy $X = ln(E_0/E)$. The collision of a neutron with a nucleus is assumed to be isotropic. Then the increase of lethargy at the collision (the decrease of energy) has a truncated exponential distribution. Finally let τ_x be the time necessary to achieve the lethargy x and r_x the length of path till the moment τ_x.

Let us now denote $F(R,\ x;\ z) = P\{r_{x;\,z} < R,\ A_{\tau x}\}$ the probability that the path of the neutron is shorter than R, provided that the neutron with initial lethargy z has not been absorbed until it achieved the lethargy x.

The first part of this paper contains the equations for the above probability and deals with the problem of uniqueness of the solution. In the second part the path-length of the neutrons is investigated.

Finally, the theory is applied to the case of a hydrogen medium and a constant (independent from the energy) nuclear cross section where simple explicit results are obtained.

(G. Németh)

MORAN, P. A. P. (Australian National Univ., Canberra) 10.9 (10.1)

The theory of some genetical effects of population subdivision—*In English*
Aust. J. Biol. Sci. (1959) **12**, 109-116 (3 refs., 1 table)

This paper considers the effects on the rate of approach to homozygosity which result from the subdivision of a population into partially isolated subgroups. Two models are discussed.

The first consists of $H+1$ subpopulations, each with N haploid individuals, such that in each group the succeeding generation contains K migrants chosen at random from the offspring of each of the remaining H subpopulations, together with $N-HK$ offspring from the original group. The generations are assumed to be non-overlapping. If, for a single locus with two possible alleles a and A, $k_{ti}(i = 1, \ldots, H+1)$ represents the number of a individuals with possible values $0, \ldots, N$ in the ith subpopulation at generation t, the process may be regarded as a Markoff chain in which the state of the system is determined by the $\{k_{ti}\}$. There are exactly two absorbing states for which the k_{ti} are all zero or all N, while from any other state all states are accessible. The matrix of transition probabilities is not easy to obtain explicitly, but if its three largest latent roots are $\lambda_0 = 1$, $\lambda_1 = 1$ and $\lambda_2(|\lambda_2| < 1)$, $|\lambda_2|$ may be found by setting up recurrence relations between the monents of the k_{ti} at generations t and $t+1$: this governs the system's asymptotic rate of approach to homozygosity. It is shown that the above subdivision into groups has only a very small effect on the drift to homozygosity when $K>2$, providing there are no selective effects.

The situation is entirely different in the second case. Here a deterministic model is considered in which the population is divided into two groups where selection operates equally but in opposite directions. Suppose that within each of the two groups, mating is at random so that zygotes aa, aA, AA occur with frequencies p_1^2, $2p_1q_1$, q_2^2 in the second subpopulations, but the relative reproductive powers are $(1+m) : 1 : (1-m)$, and $(1-m) : 1 : (1+m)$ respectively, m being a small selective advantage. If the succeeding generation in each group consists of a fraction k $(k<\frac{1}{2})$ of descendants from the other subpopulation, and $(1-k)$ of its own, it is shown that a stationary solution exists for the frequencies of the genes a and A, so that a stable dimorphism will result.

(J. Gani)

MORAN, P. A. P. & WATTERSON, G. A. (Australian National Univ., Canberra) 10.9 (10.1)

The genetic effects of family structure in natural populations—*In English*
Aust. J. Biol. Sci. (1959) **12**, 1-15 (7 refs.)

This paper deals with the rate of approach to homozygosity of dioecious diploid populations whose generations do not overlap. The probability that a finite genetic population without mutations is not totally homozygous at the tth generation $(t = 0, 1, 2, \ldots)$ is asymptotically equal to $C\lambda^t$, where λ is called the rate of approach to homozygosity and C is a constant.

For certain models of Wright [*Genetics* (1931) **16**, 97-159; *Proc. Nat. Acad. Sci., Wash.* (1933) **19**, 411-419], Moran [*Proc. Camb. Phil. Soc.* (1958) **54**, 60-71], and Watterson (unpublished data) with specific distributions of offspring per individual of the parent generation, the known values of λ are given. These results are generalised to cases with arbitrary offspring distributions, and with permanent marriage.

First, the family structure of populations of fixed size N is considered, with N_1 males and N_2 females where $N_1+N_2 = N$. The authors obtain the generating functions $P_1(Z)$, $P_2(Z)$ for offspring distributions of male and female parents respectively, in terms of the birth and death rates for each individual of specified age; the first two moments of these distributions prove to be of importance in subsequent theory. As examples, the generating functions applying to the previous population models are given.

The effect of family structure on the genetic behaviour of a population after a large number of generations is now studied; the population is assumed to be diploid, with a diallelic genetic factor. The autosomal case in a non-overlapping generation model with independent random mating between adults in each generation is considered first. It is shown how λ, the largest non-unit root of the stochastic matrix for the Markoff chain of genetic states may be evaluated from a set of difference equations as approximately

$$\lambda = 1 - N^{-2}(N_1P_1''(1) + N_2P_2''(1))/8.$$

This value remains unaffected when permanent marriage occurs, with $N_1 = N_2 = N/2$ and identical variances for both offspring distributions. Further results are stated for the sex-linked case.

Finally, though the general solution for overlapping generation models is not known, it is shown that λ is the same for a pair of models of Moran's type whether generations overlap or not. That this result does not hold in general has been shown by Watterson, using a second pair of equivalent models. The conditions required to extend results in non-overlapping to overlapping generation models require further investigation.

(J. Gani)

PHILLIPS, A. W. (London School of Economics) **10.8** (6.6)

The estimation of parameters in systems of stochastic differential equations—*In English*
Biometrika (1959) **46**, 67-76 (15 refs.)

A system of linear stochastic lagged dependences between n variables which are functions of continuous time can be described by a system of convolution integrals; the Laplace transform of the weighting function is called the transfer function. In the present paper it is assumed that:

(i) the transfer functions are all rational,

(ii) the numerator polynomials are of lower degree than the denominator polynomials (to avoid instantaneous responses),

(iii) that the system is stable,

(iv) that the disturbances are stationary stochastic variables whose covariance functions possess rational two-sided Laplace transforms.

For convenience, these disturbances are assumed to be generated (by further convolutions) from a set of variables (independent " white-noise " sources) which are not physically realisable.

Relations are established between the matrices of Laplace transforms of the various covariance functions involved; this leads to an expression giving the lagged covariances of the observed variables as a finite sum of exponential terms.

It is shown that the matrix of Laplace transforms of these covariances is completely determined by the values of the covariances at a sufficient number of points at equal intervals in time.

After rationalising, i.e. when all the defining equations are of the same order, it is shown that the structure of the whole system is completely determined by the above matrix. If all the equations are not of the same order they are not all identifiable, unless it is assumed that the disturbances are not cross-correlated.

The case of variables in discrete time is discussed briefly; it is stated that sampling properties of estimates of the continuous system may be studied through this case. Some sampling experiments are in progress.

(C. L. Mallows)

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PRABHU, N. U. (Univ. of W. Australia and Karnatak Univ., India) **10.4** (10.2)

Some exact results for the finite dam—*In English*
Ann. Math. Statist. (1958) **29**, 1234-1243 (12 refs.)

This paper derives stationary distributions Z_t in connection with the discrete finite dam model proposed by Moran [*Aust. J. Appl. Sci.*, **5**, 116-124] where the storage process $\{Z_t\}$ is known to be a Markoff chain. Stationary distributions are derived for the cases where the input is (i) geometric, (ii) negative binomial or (iii) Poisson type and the output (release) is a unit amount per unit time. In the final section is discussed the problem of emptiness in a finite dam.

The storage system considered in this paper is that of a dam (or queue) of finite capacity whose storage Z_t is defined for discrete time ($t = 0, 1, 2, \ldots$) as the dam content just after an instantaneous release at time (t) and before an input flows into it during the interval (t, $t+1$). A full description of the model, with references, was given in 1957 by Gani [*J. R. Statist. Soc.* B, **14**, 181-207].

In connection with the case of an instantaneous unit output (release) at time (t), it is shown that the relevant Markoff chain is irreducible and contains a finite number (K) of states. Therefore the stationary probability distribution exists.

The first case to be considered is where the input distribution is of the geometric type: $g_j = Pr\{X_t = j\} = ab^j$, ($j = 0, 1, 2, \ldots$) where $0<a<1$ and $b = 1-a$. It is shown that the stationary distribution is also of the geometric type, truncated at $Z = K-1$ and with a modified initial term. For the negative binomial input $g_j = n_j \binom{n+j-1}{j} a^n b^j$, with n a positive integer, the stationary distribution of dam storage is the weighted sum of n geometric distributions, Each of these is truncated at $Z = K-1$ and has a modified initial term. Thirdly, in the case of the Poisson input, $g_j = e^{-\rho}\rho^j/j!$, the author, while stating that complex variable methods should be used, uses a heuristic argument involving the Poisson as the limiting case of the negative binomial ($n\to\infty$, $a\to 1$ and $\rho = nb/a$ held fixed).

The concluding discussion is on the problem of emptiness of the finite dam: the probability where, starting with an arbitrary storage Z_0, the dam becomes empty before it overflows: in random walk terminology, the probability of absorbtion at $Z = 0$ [W. Feller (1950), *Introduction to probability and its applications*, Wiley, 300-303].

(W. R. Buckland)

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PYKE, R. (Columbia University) **10.1** (5.6)

The supremum and infimum of the Poisson process—*In English*
Ann. Math. Statist. (1959) **30**, 568-576 (11 refs.)

The author studies the behaviour of the extrema of a Poisson process with shift, i.e. the stochastic process $X(t) = Y(t) - \alpha t$ where $\alpha > 0$ and $Y(t)$ is a Poisson process with parameter $\lambda > 0$. An explicit expression is derived for the probability that in the time interval $(0, T)$ the value of the process never exceeds the constant $x (x \geqq 0)$, i.e. $P\left\{\sup_{0 \leqq t \leqq T} X(t) \leqq x\right\}$. An explicit expression is also obtained for the probability that the process value falls below the constant $x (x \leqq 0)$ at some point in the interval $(0, T)$, i.e. $P\left\{\inf_{0 \leqq t \leqq T} X(t) \leqq x\right\}$. For both problems the limiting probabilities as $T \to \infty$ are calculated, and in the latter problem the limit probability is shown to be an exponential function of x.

The results have immediate application in queueing theory. From these expressions the distribution for the length of time a queue is busy is determined explicitly for the case of Poisson input and constant service times and for the case of constant arrival times and exponential service times. Applications to the D_n^+-statistic and related distribution-free statistics are indicated.

(R. G. Miller, Jr.)

REICH, E. (University of Minnesota) **10.4** (0.4)

On the integrodifferential equation of Takács. II—*In English*
Ann. Math. Statist. (1959) **30**, 143-148 (5 refs., 1 fig.)

It will be most convenient to let this review cover the author's first paper [*Ann. Math. Statist.* (1958) **29**, 563-570] as well as the paper listed in the caption. These two papers are concerned with an integrodifferential equation that arises in the study of queueing, counter, telephone and inventory models. Suppose for description purposes that telephone calls arrive at an exchange according to a non-homogeneous Poisson process with density function $\lambda(t) \geqq 0$. That is to say, the probability that a call will arrive in the time interval $(t, t+\Delta t)$, is approximately equal to $\lambda(t)\Delta t$. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed positive random variables with common distribution function H, and let X_0 be a random variable with distribution function H_0. X_0 represents the duration time of the call which arrives at time $t = 0$ while $X_i (i = 1, 2, \ldots)$ represents the duration time of the ith call to arrive. Suppose one assumes that there is only one channel available for the incoming calls, and that each call is able to wait until the conversations of all proceeding calls are completed. It may be shown that the waiting time of a person placing a call at time t (that is, the time until he may begin his conversation) is the random variable

$$\eta(t) = \sup_{u>0} \left[\sum_{j=N(t-u)+1}^{N(t)} X_j - u \right]$$

where $N(t)$ denotes the number of calls which arrive during the time interval $(0, t)$. Set $F(t, x) = Pr[\eta(t) \leqq x]$. It was shown by Takács [*Acta Math. Hung.* (1955) **6**, 101-129] that $F(t, x)$ sztisfies the integrodifferential equation

$$(1)\quad \partial F(t,x)/\partial t - \partial F(t,x)/\partial x = -\lambda(t) \int_{0-}^{x} [1 - H(x-y)] d_y F(t,y).$$

Of particular interest is the quantity $F(t) \equiv F(t, 0)$, the probability that a call arriving at time t will be connected immediately.

In the earlier paper, the author proves, using complex variable techniques, that equation (1) uniquely determines $F(t)$ for all t (and hence $F(t, x)$ for all t and x). More precisely, he shows that under minor regularity conditions on λ and H, $F(t)$ is the unique continuous solution of a Volterra equation of the second kind. Analogous results are derived for the distribution function $B(t, x) = Pr[\eta^*(t) \leqq x]$ where $\eta^*(t)$ is the random variable defined to equal $\eta(t)$ if $\eta(u) > 0$ for all $u \in (0, t)$ and to equal zero otherwise. In particular, $B(t, 0)$ is simply the probability that the telephone exchange had an idle period before time t. In the last section of this paper, sufficient conditions are given to ensure that $\eta(t)/t \to 0$ in probability as $t \to +\infty$.

In the second paper, the author studies the asymptotic behaviour of $F(t)$ as $t \to +\infty$. He proves several preliminary Abelian lemmas, from which it follows that under a condition on the growth of the function $\Lambda(t) = \int_0^t \lambda(u)du$, one has $T^{-1} \int_0^T F(t)dt = 1 - \mu T^{-1} \Lambda(T) + o(1)$ as $T \to +\infty$ where $\mu = E(X_1)$. This reduces to a known result in the case where $\lambda(t)$ is a constant.

(R. Pyke)

RÉNYI, A. (Math. Inst., Hungarian Academy of Sciences, Budapest) **10.5** (2.0)

On the dimension and entropy of probability distributions—*In English*

Acta Math. Acad. Sci. Hung. (1959) **10**, 193-215 (4 refs.)

In a previous paper [J. Balatoni and A. Rényi, " Remarks on entropy," *Magy. Tud. Akad. Mat. Kut. Intézet. Közleményei* (1956) **1**, 9-40] the notion of the dimension and entropy of a random variable ξ were defined as follows: let $\xi_n = 1/n[n\xi]$ and $H_0(\xi_n)$ Shannon's entropy of ξ_n. If the limit $d(\xi) = \lim_{n\to\infty} H_0(\xi)/\log n$ (where log stands always for $\log_2$) exists, then $d(\xi)$ is called the dimension of ξ. If, in this expression and putting $d = d(\xi)$, the limit

$$H_d(\xi) = \lim_{n\to\infty} (H_0(\xi) - d \log n)$$

also exists, then $H_d(\xi)$ is called the d-dimensional entropy of ξ. The results of the present paper are the following: if $H_0([\xi])$ is finite then

$$\underline{d}(\xi) = \varliminf_{n\to\infty} H_0(\xi n)/\log n, \quad \overline{d}(\xi) = \varlimsup_{n\to\infty} H_0(\xi n)/\log n$$

are always finite and $0 \leqq \underline{d}(\xi) \leqq \overline{d}(\xi) \leqq 1$. That this notion of entropy is in accordance with that of Shannon is shown by the following theorem: if ξ has a density $f(x)$ and $H_0([\xi])$ is finite, then $d(\xi) = 1$ and $H_1(\xi) = -\int_{-\infty}^{\infty} f(x) \log f(x) dx$ provided the integral exists. Another theorem states that if $F(x) = P(\xi < x)$ has the representation

$$F(x) = (1-d)F_0(x) + dF_1(x)$$

where $F_0(x)$ is a purely discontinuous and $F_1(x)$ is an absolutely continuous distribution function then $d(\xi)$ exists and equals d. Further, if Shannon's entropy of both $F_0(x)$ and $F_1(x)$ exist, then $H_d(\xi)$ exists also and equals the weighted sum with weights $1-d$, d of the entropies of $F_0(x)$ and $F_1(x)$, respectively, plus the entropy of the probability distribution $\{d, 1-d\}$. The above notions and results are generalised for vector-valued random variables.

The case of a random variable with values in an abstract metric space is considered also and a theory is sketched which corresponds to the theory of the $\mathscr{E}$-entropy of Kolmogoroff. Let X be a completely bounded metric space and $X_1(\mathscr{E})$, ..., $X_n(\mathscr{E})$ a subdivision of X into non-overlapping sets, each having a diameter $\leqq \mathscr{E}$ and belonging to the σ-algebra generated by the spheres and let $N(\mathscr{E})$ be the minimal number of sets into which X can be divided in such a manner. Let further $\xi \mathscr{E} X$ be a random variable and $H(\mathscr{E}, \xi)$ the Shannon-entropy of the probability distribution $P_k(\mathscr{E}) = P[\xi^{-1}\{X_k(\mathscr{E})\}]$, $k = 1, 2, \ldots, n(\mathscr{E})$. It is supposed that $\lim_{\mathscr{E}\to 0} \log n(\mathscr{E})/\log N_x(\mathscr{E}) = 1$ and the lower and upper limits of $H(\mathscr{E}, \xi)/\log n(\mathscr{E})$ (when $\mathscr{E}\to 0$) are denoted by $\underline{D}(\xi)$ and $\overline{D}(\xi)$, respectively. If $\underline{D}(\xi) = \overline{D}(\xi)$, then this common value is called the dimension of ξ. Some consequences are drawn for the case $X = [0, 1)$.

(A. Prékopa)

SCHNEIDER, B. (Inst. für mech. Verfahrenstechnik, Karlsruhe) **10.0** (9.0)

Methods of time series analysis in biometrics—*In German*

Biom. Zeit. (1959) **3**, 162-189 (8 refs., 7 tables, 6 figs.)

In the introduction, the author points out that the theory of time series has developed mainly on the basis of problems in physics, astronomy and econometrics, and that biology, for a long time, played a secondary rôle as a stimulus and/or field of application of the theory. Then this theory was introduced to agricultural sciences, this was done in a rather empirical way. A satisfactory mathematical foundation was given in 1948 by A. Hald [*The decomposition of a series of observations*]; in the present article, the author applies the algorithm given by Hald to the practical needs of biology.

The author begins with a general survey of the theory of time series and its principal methods—following the terminology of Hald and H. Wold (1954). The two principal objectives are stated as:

(*a*) to find a hypothetical process satisfying the theoretical requirements of the phenomenon in question, and corresponding with sufficient accuracy to the given (observed) values;

(*b*) to provide a basis to enable forecasts to be made from the structure of the given time series.

The three different stochastic processes, possibly underlying the structure of a biological series of observations, are given as:

(1) a purely random process,

(2) a process composed of trend, periodic component and random component,

(3) a process of linear regression.

The main part of the article is devoted to the analysis of the " composed " process. In biological field experiments the parameter " time " is very often to be substituted by the concept of " rows " or " blocks ". If the trend-component can be regarded as a constant within a block (i.e. within the length of one period), the problem can readily be solved by analysis of variance. In most cases, however, this assumption cannot be made ; two different approaches are then feasible:

(i) to lay out the experiment in a different way, in order to achieve a constant trend component;

(ii) to eliminate the trend component, either by the method of moving averages or by a regression parabola (of the nth degree).

The author proceeds to expound these two methods in detail, limiting the analysis to moving averages of odd numbers of observations. Applying regression analysis, the estimate of the trend component is found by orthogonal polynomials. Tests for the estimates of the trend component and for the existence of " periodicity " are given for both cases with references for the proofs.

Finally, a five-by-five table of observed values is used to illustrate the use of both methods. The computation process is shown, the different tests are applied, and the results are compared.

(G. Bruckmann)

SMITH, W. L. (University of North Carolina, Chapel Hill) **10.1** (1.5)
On the cumulants of renewal processes—*In English*
Biometrika (1959) **46**, 1-29 (21 refs.)

A *renewal process* is a sequence of independent non-negative and identically distributed random variables. Writing S_k for the sum of the first k such variables, this paper is concerned with the cumulants of the random variable N_t which is defined to be the maximum suffix k for which $S_k \leqq t$. The (weak) assumption is made throughout that for some finite k the distribution of S_k has an absolutely continuous component. The main result is that, if the $n+p+1th$ moment of the renewal distribution is finite for some non-negative p, then the nth cumulant of N_t is

$$a_n t + b_n + \lambda(t)(1+t)^{-p}$$

where $\lambda(t)$ is of bounded total variation, is $O(1)$ as t tends to infinity, and has two further properties.

The proof is in several stages. First the properties of advancing factorial moments and cumulants are discussed. There are some lemmas on the Taylor expansions of certain Laplace-Stieltjes transforms; in particular it is shown that a necessary and sufficient condition for a non-negative variable to have its first k moments finite is that its Laplace-Stieltjes transform can be expanded up to the $k-1th$ term, with a remainder term involving the Laplace-Stieltjes transform of another non-negative variable.

A symbolic calculus is developed for dealing with complicated Laplace transforms; this leads to a proof that the cumulants considered are at most polynomial (of degree n) in t with the desired form of remainder. To show that all except two of the coefficients in these polynomials vanish, a complicated complex variable argument is employed; the result implies a curious combinatorial identity involving separations of multipartite numbers which is further discussed in an appendix but for which an elementary proof has not been found.

The determination of the coefficients a_n and b_n involves further combinatorial calculations which have been carried out up to $n = 8$.

(C. L. Mallows)

STUR, D. (Bundesstaatl. bakt.-serol. Unters.-anst., Vienna) **10.9** (6.8)
Virus inactivation by formaldehyde as a single-hit event—*In German*
Metrika (1959) **2**, 62-77 (16 refs., 2 tables, 3 figs.)

Virus inactivation by formaldehyde commonly yields inactivation- (survival-) curves systematically different from the usual target-theory curves, e.g. single- or multi-hit, constant, varying at random or subdivided sensitive area.

In order to obtain a better fit to the curve it is postulated that inactivation is due to a single-particle hit but that due to a tanning effect, the sensitive area decreases exponentially with time. This corresponds essentially to a stochastic immigration-emigration process for the inactivating "particles". In this case, the virus survival curve follows a double-exponential $Q(y) = \exp\{-\lambda/\mu[1-\exp(-\mu t)]\}$ with a non-zero asymptote.

The author's derivations yield more complicated expressions (exponentials with a variety of truncated geometric series parameters) arising from the use of discrete instead of continuous time. A table is given to assist in fitting the experimental curves by trial and error. The paper concludes with a discussion of the results.

(R. Wette)

TAKÁCS, L. (Math. Inst., Hungarian Academy of Sciences, Budapest) **10.1 (–.–)**
On a probability problem in the theory of counters—*In English*
Ann. Math. Statist. (1958) **29**, 1257-1263 (8 refs.)

In this paper, which is a generalisation of the problem investigated in 1953 by G. E. Albert and L. Nelson [*Ann. Math. Statist.* **24**, 9-22], it is supposed that particles arrive at the counter in the time interval $(0, \infty)$ according to a Poisson process with density (λ). Each particle arriving, independently, gives rise to an impulse with probability 0 or 1 according to whether at the particular instant there is an impulse present. Those particles which occur at an instant when there is no impulse present become "registered particles". If the number of registered particles in the interval $(0, t)$ is ν_t, what is the distribution of ν_t and its behaviour as $t \to \infty$.

After considering the structure of the process, the author gives some general theorems for ν_t and discusses the determination of the distribution function and the two states A and B, i.e. whether or not an impulse covers a particular instant (t). Finally, this paper is related, through appropriate references, to other work by the author on this particular subject.

(W. R. Buckland)

TAKÁCS, L. (Math. Inst., Hungarian Academy of Sciences, Budapest) **10.4 (1.5)**
On the limiting distribution of the number of coincidences concerning telephone traffic—*In English*
Ann. Math. Statist. (1959) **30**, 134-142 (7 refs.)

Suppose telephone calls arrive at a telephone exchange in such a way that the times between successive calls may be considered as a sequence of independent and identically distributed random variables (i.e. a renewal process). The author assumes that an infinite number of channels are available at the exchange and that each call leads to a conversation. The duration time of each conversation is assumed to be a random variable having the exponential distribution function $1-e^{-\mu x}$. The duration and the arrival times are assumed to be independent.

For each $t \geqq 0$ define the random variable $\eta(t)$ to be the number of conversations in progress at time t. Set

$$P_k(t) = Pr[\eta(t) = k] \quad \text{and} \quad B_r(t) = \sum_{k=r}^{\infty} \binom{k}{r} P_k(t) \quad \text{for}$$

$k, r = 0, 1, 2, \ldots$. The author derives explicit expressions for the Laplace transforms (over t) of these quantities in terms of the common distribution function of the inter-arrival times.

The stochastic process $\{\eta(t) : t > 0\}$ is said to be in state k at time t if $\eta(t) = k$. The author also studies the expected number, $m_k(t)$, of transitions of the process from state k to state $k+1$ in the time interval $(0, t)$, as well as the distribution function, $R_k(t)$, of the time between successive transitions of this kind. He obtains explicit expressions for the Laplace transforms (over t) of $m_k(t)$ and $R_k(t)$. The method of derivation is to construct recursive integral equations in related quantities which may then be solved using Laplace transforms. The recursive nature of these derivations relies heavily on the exponential form of the distribution function of the duration time.

(R. Pyke)

WEVER, F. (Universität Erlangen, Germany) **10.1 (10.6)**

On stationary zero-one processes with bounded gaps—*In German*

Math. Zeit. (1959) **71**, 283-288 (4 refs.)

Let $\{x_n;\ n = 0, 1, 2, \ldots\}$ be a stationary stochastic process such that each x_n takes only the values 0 and 1 with probabilities p and $1-p$, and define the "gap distribution" by

$p_\nu = \Pr\{x_{n+1} = x_{n+2} = \ldots = x_{n+\nu-1} = 0, x_{n+\nu} = 1 \mid x_n = 1\}$.

The process is called a gap process if it has the following properties considered by Akaike [*Ann. Inst. Stat. Math.*, Tokyo (1956) **8**, 87-94]:

(i) there is a number k such that $p_k > 0$ and $p_\nu = 0$ for $\nu > k$;

(ii) the greatest common divisor of the integers ν with $p_\nu > 0$ is 1;

(iii) the gap lengths are independent.

The latter condition implies recursion formulas

$$P_n = p_1 P_{n-1} + \ldots + p_n P_0$$

for the conditional probabilities $P_i = \Pr\{x_{m+i} = 1 \mid x_m = 1\}$. The average waiting time is given by $\sum_i i p_i = p^{-1}$, the covariance function by $R(n) = p(P_n - p)$ and $p = \lim_n P_n$. The author proves again the theorem by Akaike that the process has a continuous (real) spectral density function $w(\lambda)$, and then determines $w(\lambda)$ as the ratio of the squared absolute values of polynomials in $e^{2\pi i\lambda}$ of degrees $k-2$ and $k-1$ respectively: these are explicitly calculated. The proofs are based only on the above mentioned properties of the p_ν's and the relations above. Finally, the author notes that the facts about moving averages which are valid for stationary processes with a rational spectral density also hold in the present case. (See Chapter 10, section 10 of Doob, *Stochastic Processes*, New York, 1953.)

(K. Krickeberg)

WHITE, J. S. (Minneapolis Honeywell Regulator Company, Minneapolis) **10.7 (3.4)**

The limiting distribution of the serial correlation coefficient in the explosive case—*In English*

Ann. Math. Statist. (1958) **29**, 1188-1197 (7 refs.)

Several authors have studied the discrete stochastic process (x_t) in which the observed x's are related by the stochastic difference equation

$$x_t = \alpha x_{t-1} + u_t \quad t = 1, 2, \ldots, T,$$

where the u's are unobservable disturbances and α is an unknown parameter. In most articles, including the current one, it is assumed that the u's are independent and identically distributed with mean zero and unknown variance σ^2.

The statistical problem is to find some appropriate function of the x's as an estimator of α and to examine its properties. The distribution of the successive x's depends on the u's and on x_0. The author considers three distributions for x_0:

(*A*) x_0 = a constant (with probability one)

(*B*) x_0 normally distributed with zero mean and variance $\sigma^2/(1-\alpha^2)$

(*C*) $x_0 = x_T$.

Distribution (*B*) is generally most appealing, because if x_0 has this distribution and the u's are normally distributed, the process is stationary. The essence of being a stationary process is that the distribution function of the x's is not affected by a translation in time of the set of observations. Distribution (*C*), the circular case, has been proposed as an approximation to (*B*) because it is much easier to analyse.

Under distribution (*A*),

$$\mathrm{Var}\,(x_t) = \sigma^2[1 + \alpha^2 + \ldots + \alpha^{2(t-1)}].$$

Hence if $|\alpha| \geqslant 1$, this variance increases without bound as t approaches infinity. Mann and Wald considered only $|\alpha| < 1$ and showed that $\hat{\alpha} = \Sigma x_t x_{t-1} / \Sigma x_{t-1}^2$ is a least squares estimator for α and that $\hat{\alpha}$ is asymptotically normally distributed with mean α and variance $(1-\alpha^2)/T$. Rubin showed $\hat{\alpha}$ to be a consistent estimator for all α. The present author derives the distribution of $\hat{\alpha}$ when $|\alpha| \geqslant 1$. He considers the random variable $g(T)(\hat{\alpha} - \alpha)$, where the standardising function $g(T)$ is $T/\sqrt{2}$ for $|\alpha| = 1$ and is $|\alpha|\ T/(\alpha^2 - 1)$ for $|\alpha| > 1$. It is shown that the limiting distribution of $g(T)(\hat{\alpha} - \alpha)$ is Cauchy for $|\alpha| > 1$ if $x_0 = 0$; for $x_0 = c$, this distribution function is much more complicated, involving an infinite series. A generating function is available for this distribution when $|\alpha| = 1$.

(W. J. Dixon)

ZÍTEK, F. (Math. Inst., Czech. Academy of Sciences, Prague) **10.0 (1.5)**

A remark on the (*BB*)-integral—*In Czech*

Cas. pěst. math. (1959) **9**, 83-89 (6 refs.)

The notion of the (*BB*)-integral has already been introduced by the author in his earlier paper " Fonctions aléatoires d'intervalle " [*Czech. Math. J.* (1958) **8**, 583-609]. In this present paper he gives a new condition for the (*BB*)-integrability of a random function of an interval, using some recent results of H. Bergström given in his papers entitled " On the limit theorems for convolutions of distribution functions " [*J. reine angew. Math.* (1957) **198**, 121-142, and (1958) **199**, 1-22].

The author first introduces the concept of uniform convergence of the Burkill integral of a real or complex (non-random) function depending on a real parameter: the corresponding finite sums must converge uniformly to their limit—the Burkill integral—in respect to the parameter. It is shown in a lemma that this uniform convergence is a fundamental property of the interval of integration.

Let us have a continuous random function of interval with the property that the upper Burkill integral of the truncated first moment of the corresponding random variables is finite for every finite interval of truncation. Let $H_\sigma(I, x)$ be the real function of an interval defined as the Weierstrass transform (convolution with the normal distribution function with mean zero and standard deviation σ) of the difference of the distribution function of the random variable associated with the interval I and the distribution function having all its mass concentrated in the origin. Then the necessary and sufficient condition for the (*BB*)-integrability of the random function considered is that, for each positive σ, the Burkill integral of $H_\sigma(I, x)$ converge uniformly in respect to all x on the whole real axis. The main idea of proving the validity of this condition is the equivalence—quoted by Bergström—of the Cauchy convergence in the W-norm (to which our condition corresponds) and the QM-convergence corresponding analogously to the conditions of (*BB*)-integrability given in the earlier paper by Zítek.

It is further shown, making use of the fundamental property of the uniform convergence of Burkill integrals and of the method of ψ-functions (logarithm of the characteristic function), that for continuous random functions of interval the (*BB*)-integrability in a given interval is equivalent to the existence of the (*BB*)-integral over this interval —this property does not hold generally. This was shown in the cited paper by Zítek.

(F. Zítek)

KENDALL, M. G. (London School of Economics) 11.0 (–.–)

Modern statistics in business and commerce—*In German*

Metrika (1958) **1**, 223-238.

In his lecture, delivered early in June 1957 for the occasion of " British Week " at the University of Munich, Professor Kendall gave a broad view of the rôle which statistics play nowadays in its application to practical economic questions. In particular, he stressed the personal side, i.e. the position in which the statistician in business and commerce is likely to find himself, and the " human relations "—problems he has to solve in pursuing his work. Professor Kendall makes clear that the modern statistician's job often consists in the very opposite of urging people to provide numerical information; a considerable part of his efforts is spent in trying to persuade them not to collect (unnecessary) information, to cut down records, simplify forms and rely rather on intelligent sampling instead of on complete enumerations. Furthermore, the statistician may very often not be able to settle an argument, but to focus it.

Regarding prediction and stochastic processes, Professor Kendall pointed out that the stochastic way of thinking and of expressing facts is something many employers of statisticians have yet to learn in order to " use " statisticians properly—provided always that statisticians learn to have courage in their forecasts.

As businesses become larger, more complicated and more monopolistic, it becomes increasingly difficult to measure their efficiency—or to accomplish maximum efficiency by the light of nature alone. What is important is not so much to determine the actual gain or loss per annum, comparing it, may be, with that of previous years, but to find out whether the business in question is as good as it might be. Statements of this kind require the application of Linear Programming and Operational Research. In theory both of these are mathematical subjects but in practice they are highly statistical ones. Firstly, because a great deal of the raw data is statistical and only a statistician can tell whether it is worth while using elaborate analytical techniques on them, and secondly, because statistical methods may have to be used to solve the purely mathematical problems. A statistical problem of extreme interest, hitherto unsolved, is that of the optimum size of a group or of a particular branch of industry, where both seem to follow some statistical law of distribution.

The supply of academically trained statisticians cannot keep pace with the demand for them, as in the last few decades statistics has entered practically every aspect of life. As Professor Kendall remarked, " there is scarcely any pie into which statisticians may not be invited to put a finger ".

(G. D. Bruckmann)

PERKAL, J. (Math. Inst., Polish Acad. Sci., Wroclaw) 11.0 (9.2)

Two new methods of analysing a collection of attributes—*In English*

Bull. Acad. Sci., III (1959) **7**, 63-66

Two new methods of analysing a collection of attributes are presented: a single-vector method and a multiple-vector method. These methods are similar to the single-factor method (of Spearman) and to the multiple-factor method (of Thurstone). Instead of factors, which are mutually orthogonal and uncorrelated with the residuals, the author proposes vectors received by linear calculations from the initial attributes. The vectors may be correlated one to another and also with the residuals. The proposed methods (unlike the factor method) are easy and simple in calculation, univalent, and the vectors have a clear meaning in natural sciences.

(L. Kubik)

QUENOUILLE, M. H. (London School of Economics) **11.1** (11.7

Tables of random observations from standard distributions—*In English*
Biometrika (1959) **46**, 178-202 (7 refs., 22 pages of tables)

One thousand random observations, arranged in twenty groups of fifty, are provided from each of eight standard distributions. The basic data have been drawn from the first two pages of Wold's (1954) " Random Normal Deviates " and are given in the first column, x_1, of each page. On each line the observations from the seven other distributions are obtained by transformation of the value of x_1 for that line. The columns are:

x_1 random normal deviates

x_2 random rectangular deviates

x_3 ransom deviates from a distribution whose logarithm was normally distributed

x_4 random deviates from an exponential distribution

x_5, x_6, x_7 random deviates from Edgeworth type A expansions. The three columns correspond to different combinations of the third and fourth cumulants which are put equal to 0 or 0.5.

x_8 random deviates from the two-sided exponential distribution.

All deviates are given to two places of decimals. The deviates x_5, x_6, x_7 were calculated using Cornish-Fisher expansions which gave sufficient degree of accuracy within the central range of observations and correlations have been made to three extreme values of x_5. Tables of the three functions of x_1 used to derive these deviates are given.

The theoretical means and variances of all the distributions were zero and unity and the overall observed values, which are given, show close agreement with these values. Grouped frequency tables of the eight sets are also given.

(N. W. Please)

Editorial Note: see also abstract No. 194.

WELCH, B. L. (University of Leeds) **11.9** (3.1)

' Student ' and small sample theory—*In English*
J. Amer. Statist. Ass. (1958) **53**, 777-788 (14 refs.)

This paper reviews the importance of the contributions of W. S. Gosset (*Student*) on the 50th anniversary of the publication of the papers, " The probable error of the mean " and " The probable error of the correlation coefficient ". The emphasis is on the relation of Gosset's work to that of others rather than on the main content of the two papers.

The author recalls that prior to Gosset's work, scientific writers using Gauss's method of least squares customarily reported a probable error with each estimated constant. Apparently there was not a great deal of concern over the generally recognised fact that only a rough probability interpretation could be associated with the limits *estimate* + *probable error* without an application of the somewhat arbitrary Bayes-Laplace method of inverse probability.

Attention is drawn to a little known 1883 paper of Edgeworth in which a result strikingly similar to Gosset's on the distribution of t is obtained by a Bayes-Laplace argument using prior density $C\sigma^{-2}d\mu d\sigma$. This work was not pursued further, apparently owing to the sensitivity of the conclusion to the prior density; and the work was not known to Gosset.

In view of the impetus towards abandonment of the Bayes-Laplace method provided by Gosset, interest naturally turns to Gosset's own views on inverse probabilty, and the author finds that the evidence does not point consistently in one direction. One finds in the paper on the correlation coefficient a discussion of appropriate assumptions for a prior distribution of ρ; but in the paper on the mean there is no mention of prior distributions; and finally, in later applications the language nevertheless suggests an interpretation in the Bayes-Laplace sense. The author reminds us that in assessing Gosset's work one must remember that he himself did not claim to be interested in the theory of inference, but rather in the solution of distributional problems.

Effects of non-normality are considered. It is recalled that Bartlett has shown that non-normality distorts the t distribution to the order $1/\sqrt{n}$ whereas the discrepancy between a t and a normal variate is only of order $1/n$. Thus the improvement in accuracy provided by the *Student* theory is quite sensitive to non-normality of the parent distribution.

In conclusion, after noting that questions regarding the logic of inference persist to the present, the author states that " we can still unreservedly commemorate in Gosset a man who played an outstanding part in contributing to our understanding of these questions ".

(R. J. Buehler)